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Spectral-domain IFS analysis in Electromagnetics

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Introduction

- Many guided and radiative E.M. phenomena are studied in the "modal" $\boldsymbol{\kappa}$-space: e.g. guided modes or crystal lattices, radiation patterns, etc.

- Whenever [*pre*-]fractal geometries generated via IFSs are concerned, spatial Fourier transforms $\mathbf{r} \leftrightarrow \boldsymbol{\kappa}$ can be easily computed, sometimes even closed-form.

- An IFS is an (infinite) iteration of similarities whose Fourier transforms are still similarities in complex 3-space $\mathbb{C}^3$, e.g. $\mathbf{r}-\mathbf{r}_0 \leftrightarrow e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}_0}$ or $\alpha\,\mathbf{r} \leftrightarrow \boldsymbol{\kappa}/\alpha$.

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Brief review on Iterated Function Systems (IFSs)

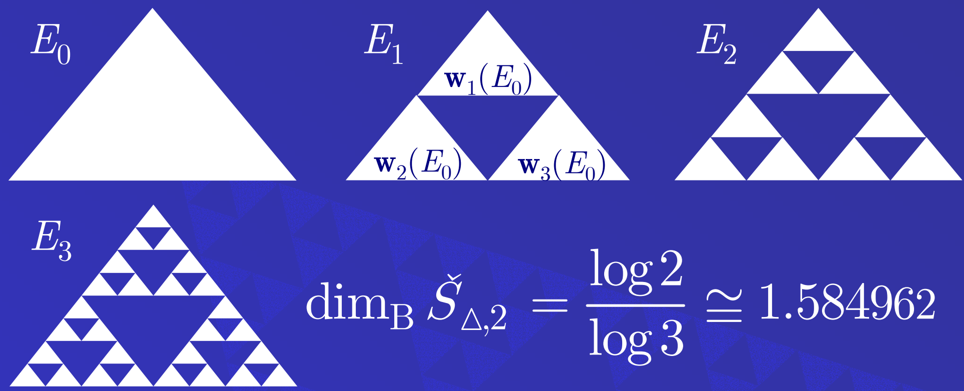
An IFS is a set of p contraction mappings $\mathbf{w}_j$: $\mathbb{R}^d \to \mathbb{R}^d$ (ratios $c_j \in]0,1[$), $1 \leq j \leq p$ such that from an initiator set $E_0 \in \mathbb{R}^d$ N^{th}-step prefractal E_N can be generated:

$$E_N = \mathbf{w}(E_{N-1}) := \bigcup_{j=1}^{p} \mathbf{w}_j(E_{N-1})$$

Limit set $E_\infty = \lim_N E_N$ is self-similar and does not depend on E_0. Its fractal dimension 'dim$_\mathrm{B}$' is often non-integer.

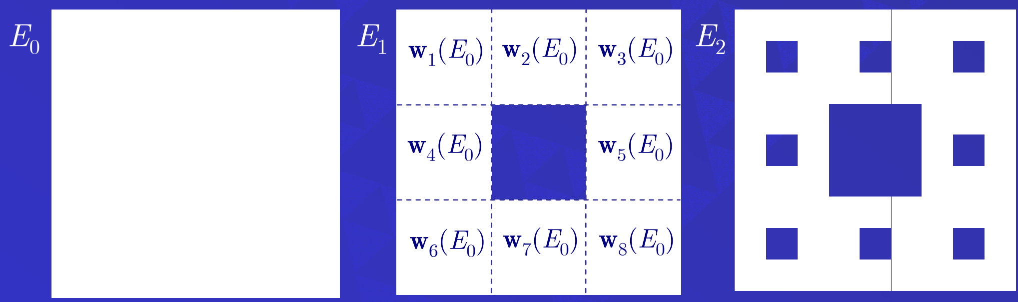
$$E_\infty = \lim_N E_N := \bigcap_{N=0}^{\infty} E_N$$

Šerpinskij gasket prefractals (p=3)



$$\dim_\mathrm{B} \check{S}_{\Delta,2} = \frac{\log 2}{\log 3} \cong 1.584962$$

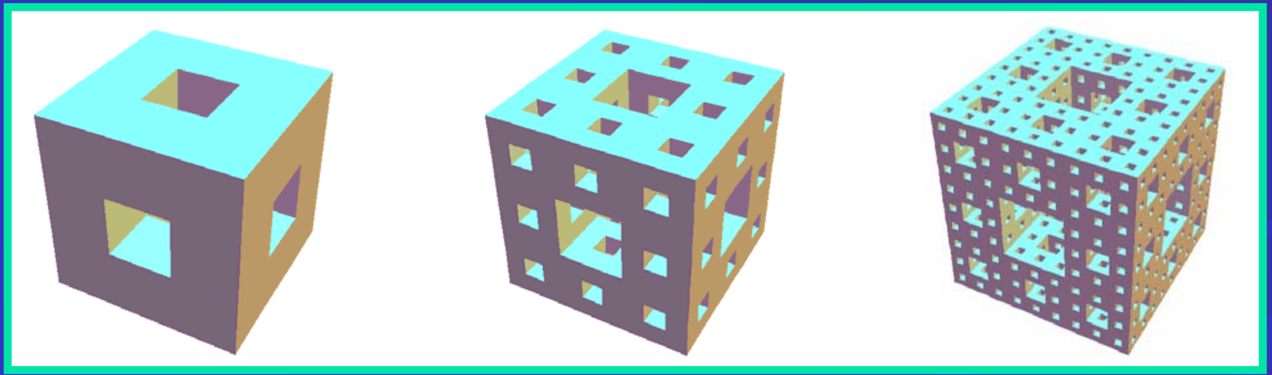
Šerpinskij carpet prefractals (p=8)



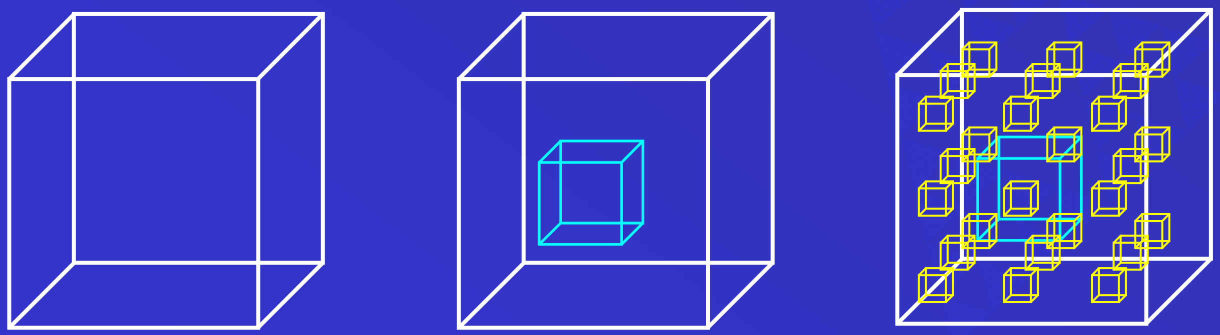
$$\dim_\mathrm{B} \check{S}_2 = \frac{\log 8}{\log 3} \cong 1.892789$$

Šerpinskij and Menger sponges

A Menger sponge (p=20) of cubic initiator set. Prefractals' copies get shrinked by 1/3 at each new iteration. It is linearly multiply connected too.



Šerpinskij sponge (p=26) $\check{S}_3$ is the "topological dual" to Menger sponge, i.e. it is superficially multiply connected: no "tunnels" inside the domain, but $(26^N-1)/25$ "holes", sank into the N^{th} prefractal.



Spectral domain radiation integrals

Vector potential $\mathbf{A}(\mathbf{r})$ for radiation from an aperture $\Omega\subset\mathbb{R}^2$ is the space convolution between aperture current densities $\mathbf{J}(\mathbf{r})=\mathbf{j}(x,y)\,\delta(z)$ and the (spatial) Green's function $g(\mathbf{r})$.

TRICK: Integration on the aperture Ω is equivalent to integrating on the whole $\mathbb{R}^2$ plane the function times Ω's characteristic function $\chi_\Omega(\mathbf{r})$, which is definied as 1 for $\mathbf{r}\in\Omega$ and 0 for $\mathbf{r}\notin\Omega$:

$$\boxed{\mathbf{A}(\mathbf{r}) = (\,g \circledast \mathbf{J}\,)(\mathbf{r}) = \frac{1}{4\pi}\iint_\Omega \frac{e^{-i\boldsymbol{\kappa}_0\cdot(\mathbf{r}-\mathbf{r}')}\,\mathbf{j}(\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|}\,\mathrm{d}\mathbf{r}' = (\,g \circledast \mathbf{j}\chi_\Omega\,)(\mathbf{r})}$$

Spectral vector potential $\boldsymbol{A}(\boldsymbol{\kappa})=\mathcal{F}[\mathbf{A}(\mathbf{r})]$ is the product of spectral current densities $\boldsymbol{J}(\boldsymbol{\kappa})$ times the spectral Green's function;

$\hat{g}(\boldsymbol{\kappa}) = \left(\kappa_0^2 - \boldsymbol{\kappa}\cdot\boldsymbol{\kappa}\right)^{-1}$. Case of a constant feeding $\mathbf{J}(\mathbf{r})=\mathbf{j}_0\chi_\Omega(x,y)\,\delta(z)$ is:

$$\boxed{\boldsymbol{A}(\boldsymbol{\kappa}) = \hat{g}(\boldsymbol{\kappa})\boldsymbol{J}(\boldsymbol{\kappa}) = \mathbf{j}_0\,\frac{\hat{\chi}_\Omega(\boldsymbol{\kappa})}{\kappa_0^2 - \|\boldsymbol{\kappa}\|^2}}$$

Šerpinskij sponges: $\boldsymbol{\kappa}$-space self-similarity

A [prefractal] aperture/antenna radiation pattern is the Fourier transform of χ_{Ω} computed for the wave vector at the (e.g.) spheric shadow boundary, i.e.: $\propto \ \|\mathbf{A}(\boldsymbol{\kappa}(\theta,\varphi))\|^2$.

For a [pre-]fractal aperture the *IFS' spatial-domain similiarities become spectral-domain's similarities.*

For $\check{S}_3$ (e.g. crystal lattice or radiating element/array shaped like a Šerpinskij sponge's N^{th} prefractal), cube-based characteristic function is a sum of 3D "rect" functions. Its Fourier transform $\hat{\chi}^{N}_{\check{S}_3}(\boldsymbol{\kappa})$ is a sum of complex-rotated 3D "sinc" functions:

$$\hat{\chi}^{N}_{\check{S}_3}(\boldsymbol{\kappa}) = \operatorname{sinc}\frac{\boldsymbol{\kappa}}{2} - \frac{1}{27}\operatorname{sinc}\frac{\boldsymbol{\kappa}}{6} - 2\sum_{k=2}^{N} 3^{-3k}\operatorname{sinc}\left(\frac{\boldsymbol{\kappa}}{2\cdot 3^k}\right) \sum_{\mathbf{c}\in\frac{(2\mathbb{N}+1)^3}{2\cdot 3^k}\cap\check{S}_{3,k-1}} \cos(\mathbf{c}\cdot\boldsymbol{\kappa})$$

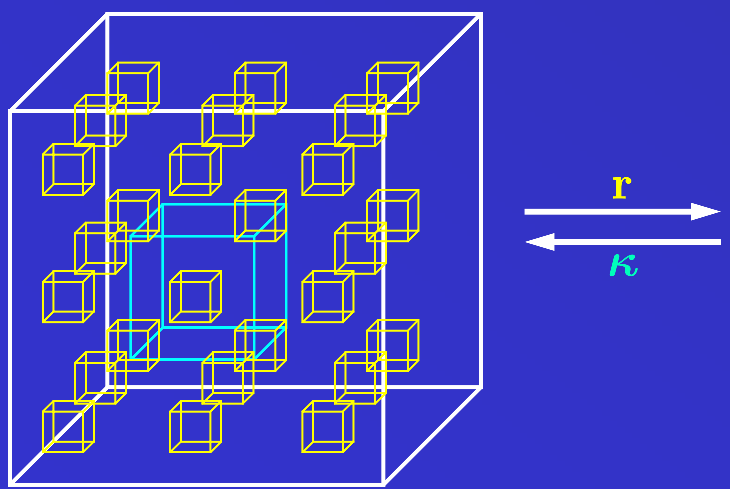
$\hat{\chi}^{N+1}_{\check{S}_3}(\boldsymbol{\kappa})$ can be iteratively built starting from previous-iterate $\hat{\chi}^{N}_{\check{S}_3}(\boldsymbol{\kappa})$:

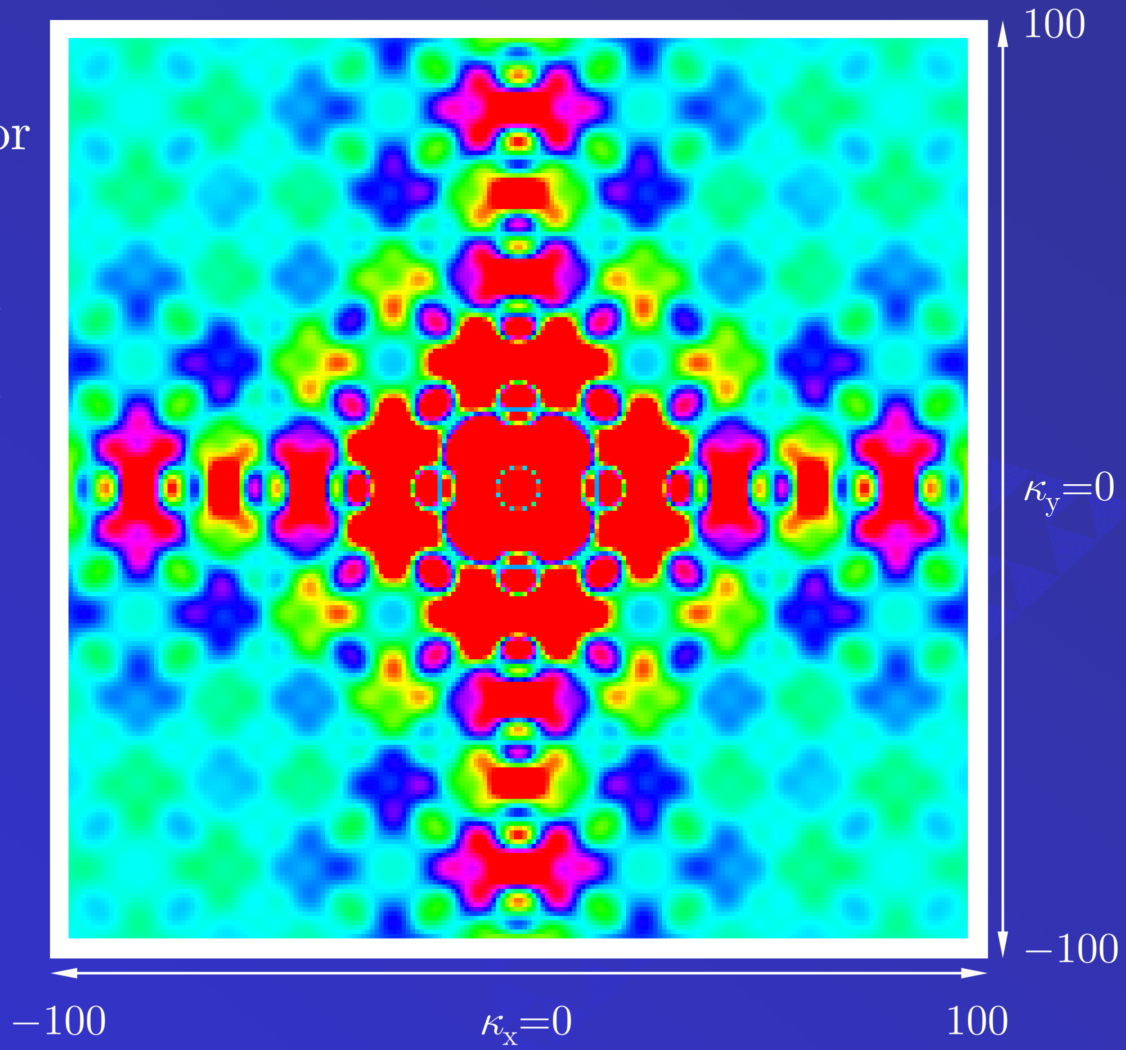
$$\boxed{\hat{\chi}^{N+1}_{\check{S}_3}(\boldsymbol{\kappa}) = \frac{1}{3}\,\hat{\chi}^{N}_{\check{S}_3}\left(3^{-1}\boldsymbol{\kappa}\right) \sum_{\mathbf{h}\in\frac{1}{2}\{-1,1\}^3} e^{-\imath\mathbf{h}\cdot\boldsymbol{\kappa}}}$$

Šerpinskij sponges: κ-space self-similarity

Slices of $\hat{\chi}^{N}_{\check{S}_3}(\boldsymbol{\kappa})$, $\boldsymbol{\kappa}=(\kappa_{\mathrm{x}},\kappa_{\mathrm{y}},\kappa_{\mathrm{z}})$ for fixed values $0\leq\kappa_{\mathrm{z}}\leq 100$.

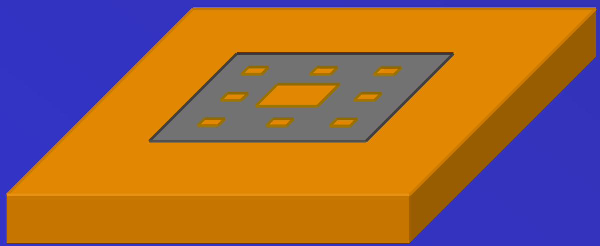
IFS-generated self-similarity is well reflected into $\boldsymbol{\kappa}$-space self-similarity: translations of "rect" pulses in $\mathbb{R}^3$ become Complex rotations of "sinc" pulses.

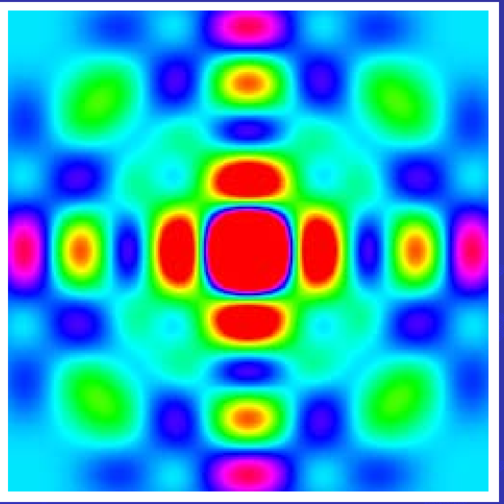




A trivial application

- Šerpinskij patch antenna $\check{S}_2$ (e.g. 3rd iteration) whose far-field radiation pattern can be estimated, or (ideal case) exactly closed-form computed in the case of constant aperture feeding.





- Artificial crystal lattice whose atoms/ions are placed into a Šerpinskij sponge's holes (with "masses" proportional to holes' ones). $\chi_{\check{S}}$'s Fourier transform is proportional to band-energy distributions $E(\boldsymbol{\kappa})$. $\hbar\boldsymbol{\kappa}$ is the tight-bound electron's momentum.

$$\boxed{E(\boldsymbol{\kappa}) = h\nu_{\mathrm{eq}}(\boldsymbol{\kappa}) = \tfrac{1}{2}\, m_{\mathrm{eq}}^{-1}(\boldsymbol{\kappa})\,\|\hbar\boldsymbol{\kappa}\|^2}$$

Conclusions

- Most methods for Electromagentics can still be used for *pre*fractal geometries by exploiting the simple nature of IFSs to overcome geometric problems.

- Singular integrals due to the Green's function cannot be simplified with methods like the one presented here.

- There are still simple transformation for the IFS in the spectral domain, which are the Fourier transform of those into the spatial domain ($\mathbf{r} \leftrightarrow \boldsymbol{\kappa}$).

- Spatial-domain self-similarities become multifractals into the spectral $\boldsymbol{\kappa}$-space (e.g. radiation patterns).