European School of Antennas
"High-frequency Techniques and Travelling-wave antennas"

TWO-DIMENSIONAL OPEN WAVEGUIDES spectral properties and leaky modes

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SUMMARY

- Introduction to the continuous spectrum of open 2D waveguides

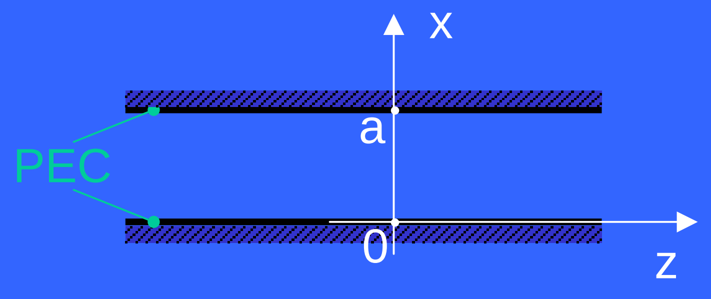
- Line-source excitation of a grounded dielectric slab (GDS); singularities of the spectral Green's function

- Steepest-Descent-Plane (SDP) representation

- Leaky modes and far-field patterns

INTRODUCTION TO THE CONTINUOUS SPECTRUM OF OPEN 2D WAVEGUIDES

CLOSED WAVEGUIDES: THE PPW (1)



Parallel-Plate Waveguide (PPW)

$$E_y(x,z) = e_y(x)\, e^{\mp jk_z z}, \quad (\mathrm{TE})$$

$$H_y(x,z) = h_y(x)\, e^{\mp jk_z z}, \quad (\mathrm{TM})$$

$$\psi = e_y, h_y$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k_x^2\psi = 0, \quad x \in [0,a]$$

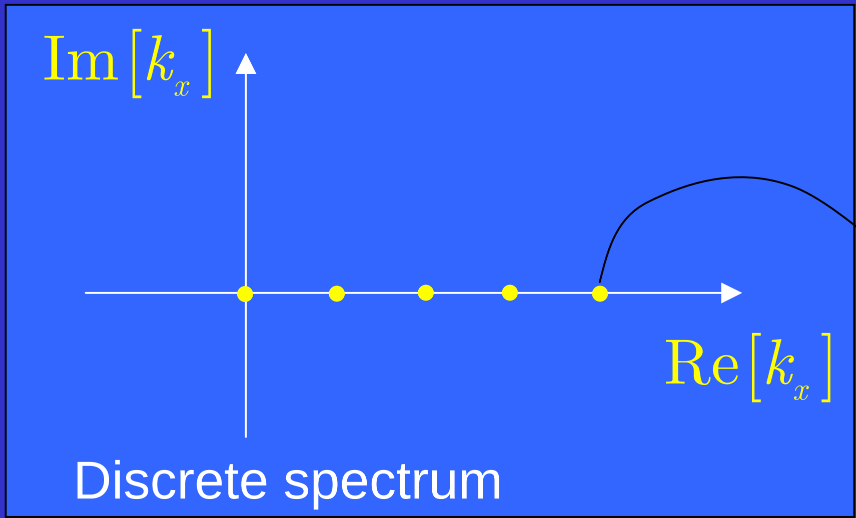
$$\psi = 0, \quad x = 0, a \qquad (\mathrm{TE})$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = 0, \quad x = 0, a \quad (\mathrm{TM})$$

$$k_x^2 = k_0^2 - k_z^2$$

(Transverse eigenvalue)

CLOSED WAVEGUIDES: THE PPW (2)

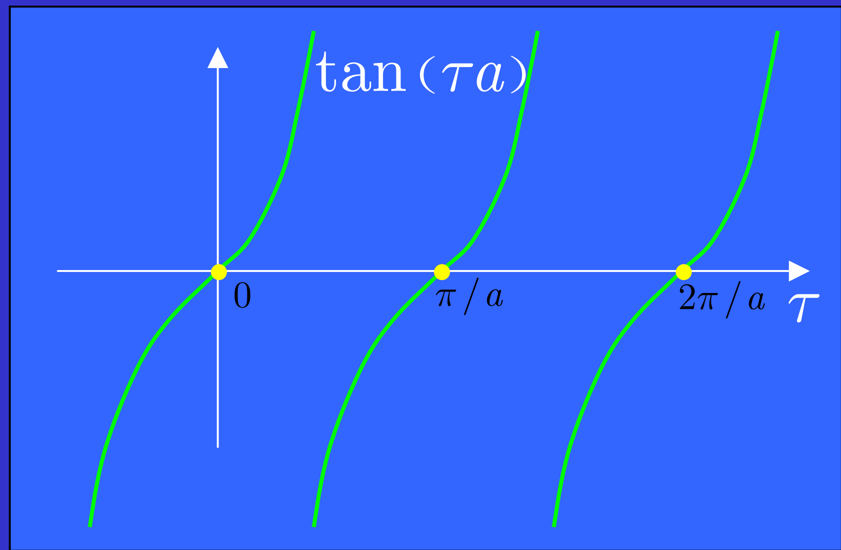


Discrete spectrum

Characteristic equation:

$$\tan(k_x a) = 0$$

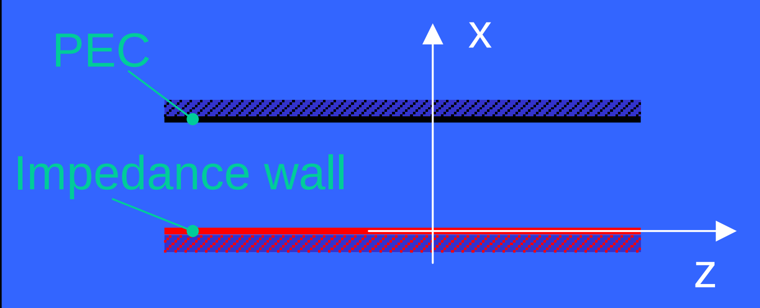
$$k_x^2 = \left(\frac{n\pi}{a}\right)^2$$



The relevant transverse operator is self-adjoint and positive semidefinite: The eigenvalues are real and non-negative

$$\psi = \begin{cases} \sin\left(\dfrac{n\pi}{a}x\right), \; n = 1, 2, \ldots \; (\mathrm{TE}) \\ \cos\left(\dfrac{n\pi}{a}x\right), \; n = 0, 1, \ldots \; (\mathrm{TM}) \end{cases}$$

CLOSED WAVEGUIDES: THE PPW (3)



PPW with one impedance wall

$$\frac{d^2\psi}{dx^2} + k_x^2\psi = 0, \quad x \in [0, a]$$

$$\psi = 0, \quad x = a \qquad (\mathrm{TE})$$

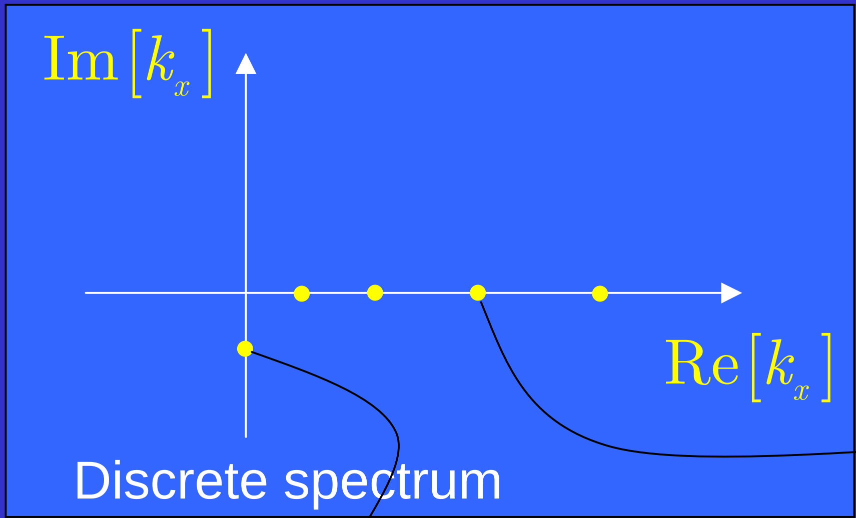
$$\frac{d\psi}{dx} = 0, \quad x = a \quad (\mathrm{TM})$$

$$\frac{d\psi}{dx} + \gamma\psi = 0, \quad x = 0$$

$$\longrightarrow \quad \gamma = \begin{cases} \dfrac{-jk_0 Z_S}{Z_0^{TE}}, & (\mathrm{TE}) \\ \dfrac{-jk_0 Z_0^{TM}}{Z_S}, & (\mathrm{TM}) \end{cases}$$

$$\text{Hyp: } \mathrm{Re}[Z_S] = 0 \Rightarrow \mathrm{Im}[\gamma] = 0$$

CLOSED WAVEGUIDES: THE PPW (4)

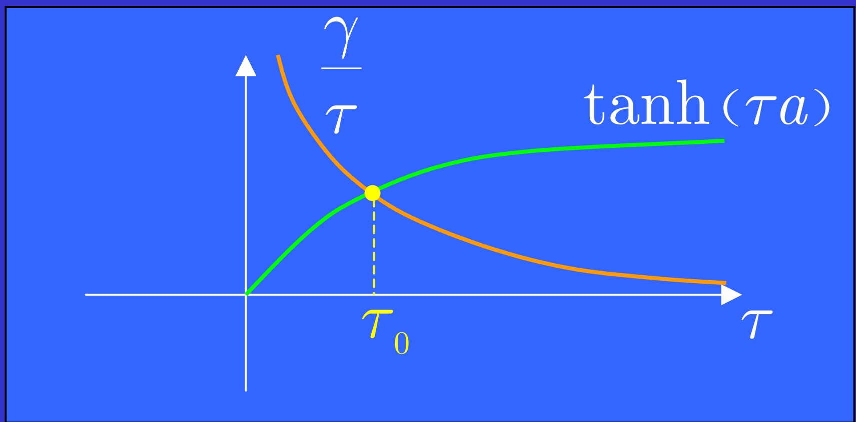


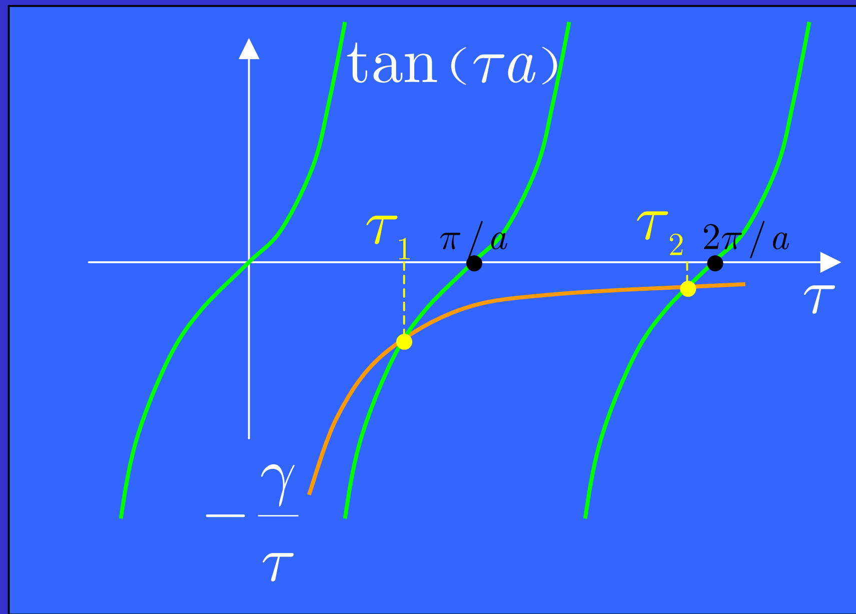
Characteristic equation:

$$\tan(k_x a) = -\frac{\gamma}{k_x}$$

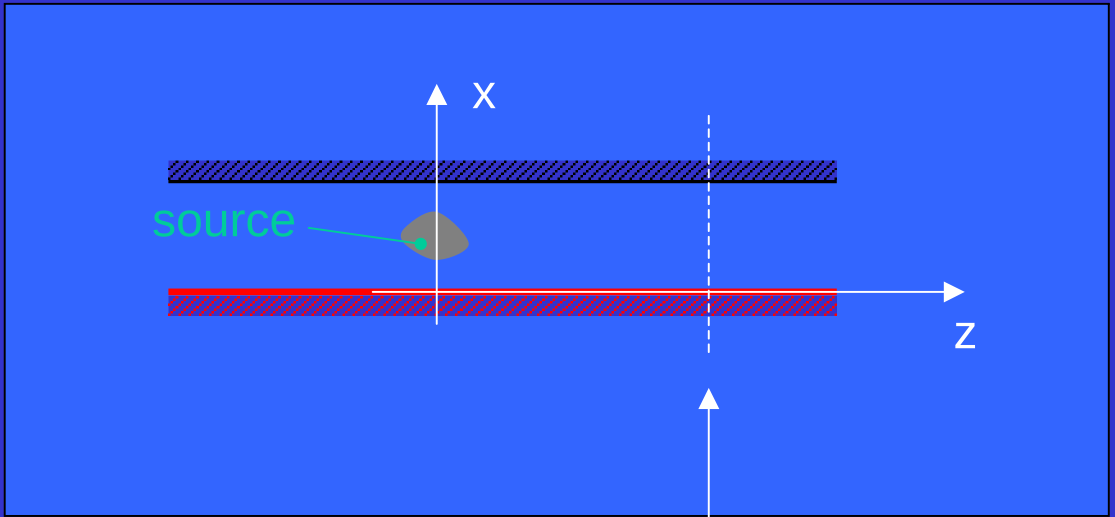
$$k_x = \tau_n (< n\pi / a)$$
$$\psi_n = \tau_n \cos(\tau_n x) - \gamma \sin(\tau_n x)$$

$$k_x = -j\tau_0$$
$$\psi_0 = e^{-\tau_0 x}$$





CLOSED WAVEGUIDES: THE PPW (5)



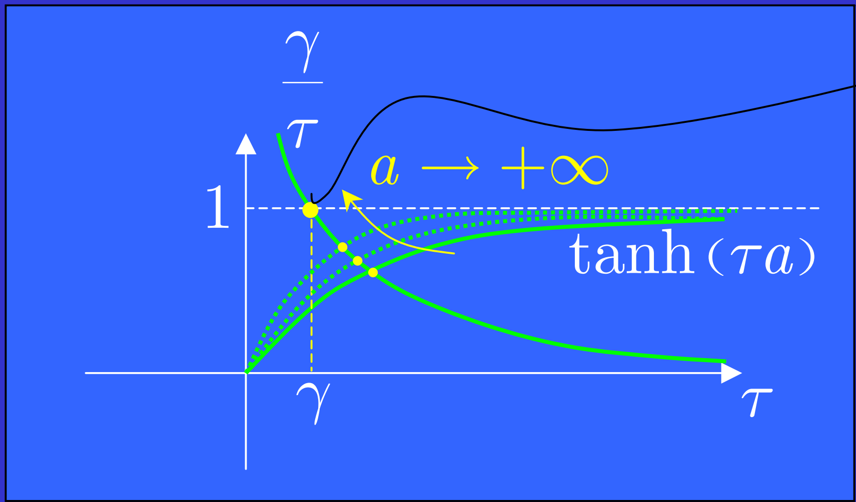
Modal representation of the field excited by a given source:

$$\begin{Bmatrix} E_y(x,z) \\ H_y(x,z) \end{Bmatrix} = A_+^{(0)} \psi_0(x) e^{-jk_{z0}z} + \sum_{n=1}^{+\infty} A_+^{(n)} \psi_n(x) e^{-jk_{zn}z}$$

$$k_{z0} = \sqrt{k_0^2 - k_{x0}^2} = \sqrt{k_0^2 + \nu^2}$$

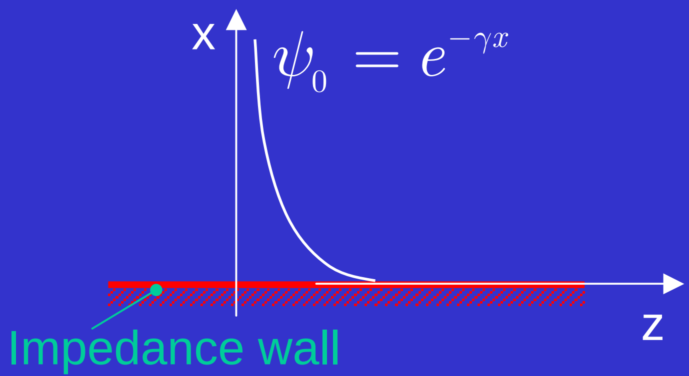
$$k_{zn} = \sqrt{k_0^2 - k_{xn}^2} = \sqrt{k_0^2 - \tau_n^2}$$

CLOSED-TO-OPEN TRANSITION (1)



$$\tau_0 \rightarrow \gamma \quad \Rightarrow \quad k_x \rightarrow -j\gamma$$

Proper eigenfunction:



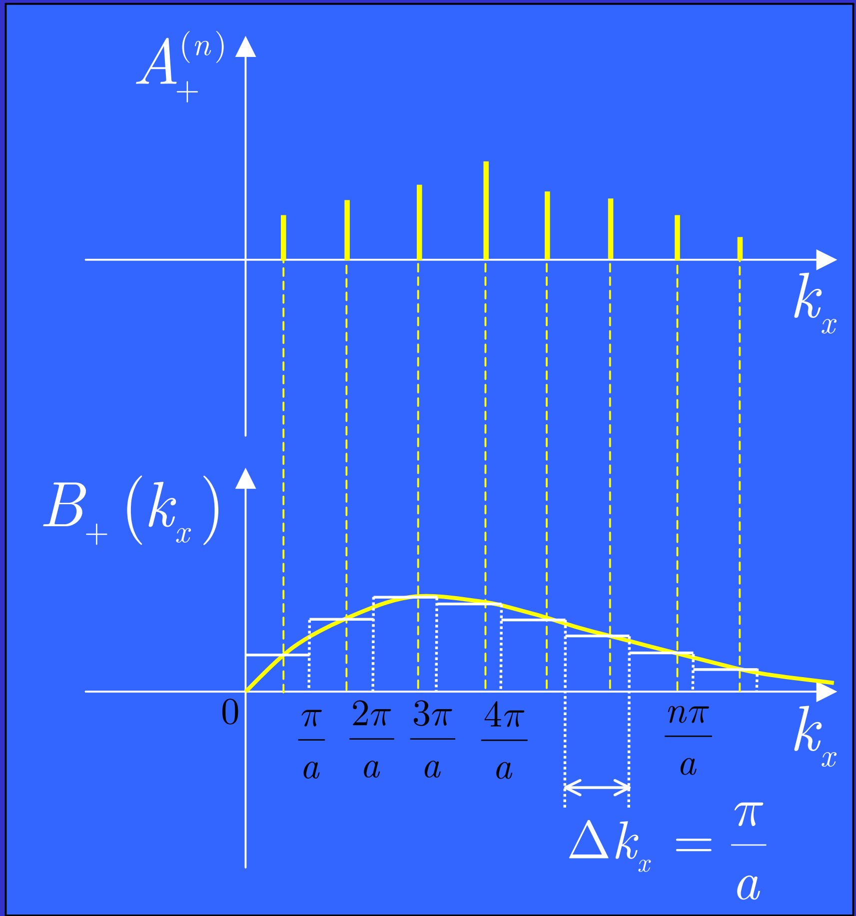
$\Rightarrow$

Surface wave of the open waveguide

$$\begin{Bmatrix} E_y(x,z) \\ H_y(x,z) \end{Bmatrix} = e^{-\gamma x} e^{-jk_{z0} z}$$

$$k_{z0} = \sqrt{k_0^2 + \gamma^2}$$

CLOSED-TO-OPEN TRANSITION (2)



$$\sum_{n=1}^{+\infty} A_+^{(n)} \psi_n(x) e^{-jk_{zn}z} =$$

$$= \sum_{n=1}^{+\infty} B_+(k_{xn}) \psi(k_{xn}, x) e^{-jk_z(k_{xn})z} \Delta k_x$$

$$a \to +\infty$$

$$\int_0^{+\infty} B_+(k_x) \psi(k_x, x) e^{-jk_z(k_x)z} \mathrm{d}k_x$$

Generalized eigenfunction of the open waveguide
(belongs to the continuous spectrum,
and is bounded at infinity)

THE CONTINUOUS SPECTRUM (1)

Generalized eigenfunction: transverse behavior

$$\boxed{\psi(k_x, x) = k_x \cos(k_x x) - \gamma \sin(k_x x) \qquad k_x \in \Re}$$

$$= \frac{1}{2}\left[(k_x - j\gamma)\, e^{-jk_x x} + (k_x + j\gamma)\, e^{+jk_x x}\right] =$$

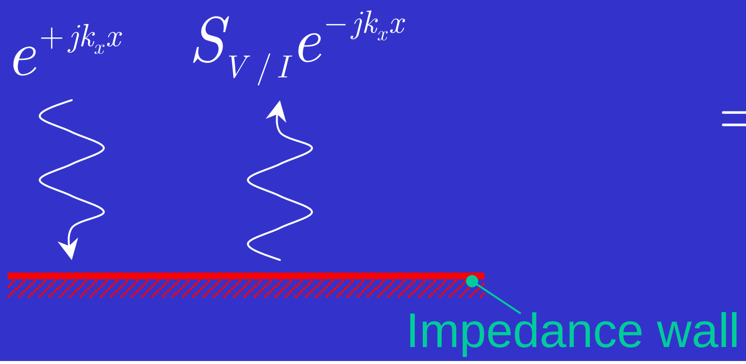
$$= \frac{1}{2}(k_x + j\gamma)\left[e^{+jk_x x} + \underbrace{\frac{(k_x - j\gamma)}{(k_x + j\gamma)}}\, e^{-jk_x x}\right] =$$

Incident wave

Reflected wave

$$= \frac{Z_S \mp Z_0^{TE/TM}}{Z_S \pm Z_0^{TE/TM}} = S_{V/I}$$

Reflection coefficient



THE CONTINUOUS SPECTRUM (2)

Generalized eigenfunction: longitudinal behavior

$$\boxed{\left\{\begin{matrix} E_y(k_x, x, z) \\ H_y(k_x, x, z) \end{matrix}\right\} = \psi(k_x, x)\, e^{-jk_z z} \qquad k_x \in \Re}$$

$$k_x < k_0 \quad \Rightarrow \quad k_z = \sqrt{k_0^2 - k_x^2} \in \Re$$

Radiative part
of the continuous spectrum

$$k_x > k_0 \quad \Rightarrow \quad k_z = -j\sqrt{k_x^2 - k_0^2} \in \Im$$

Reactive (evanescent) part
of the continuous spectrum

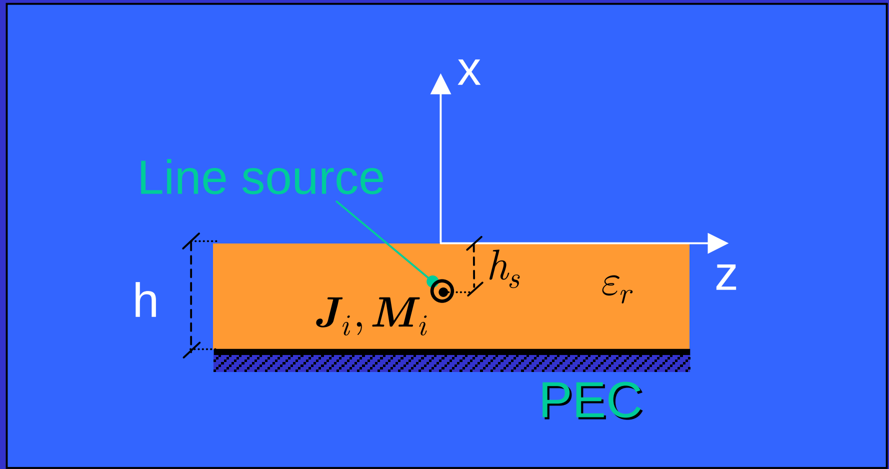
The eigenfunctions in the radiative part allow to describe radiation from sources in the presence of open waveguides

The eigenfuntions in the evanescent part allow to represent energy storage in the vicinity of such sources (like modes below cutoff in closed waveguides)

LINE-SOURCE EXCITATION

OF A GROUNDED DIELECTRIC SLAB

GDS EXCITED BY A LINE SOURCE

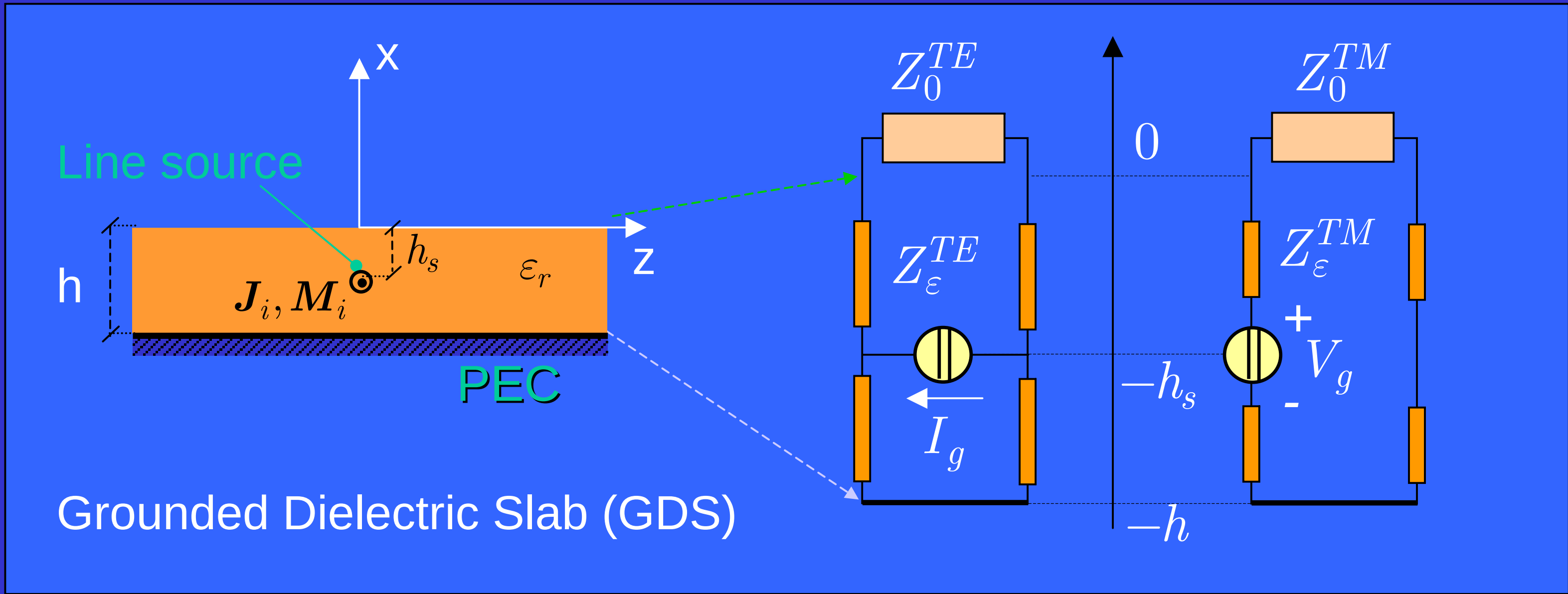


Field at the air-slab interface: representation as an inverse Fourier transform:

$\boldsymbol{J}_i$ ➡ TE $$E_y(z) = \frac{1}{2\pi} \int_{\Re} \tilde{E}_y(k_z) e^{-jk_z z} \mathrm{d}k_z$$

$\boldsymbol{M}_i$ ➡ TM $$H_y(z) = \frac{1}{2\pi} \int_{\Re} \tilde{H}_y(k_z) e^{-jk_z z} \mathrm{d}k_z$$

SPECTRAL GREEN'S FUNCTION (SGF) (1)



$$Z_0^{TE} = \frac{\eta_0 k_0}{k_{x0}} \qquad Z_0^{TM} = \frac{\eta_0 k_{x0}}{k_0} \qquad k_{x0} = \sqrt{k_0^2 - k_z^2}$$

$$Z_\varepsilon^{TE} = \frac{\eta_0 k_0}{k_{x\varepsilon}} \qquad Z_\varepsilon^{TM} = \frac{\eta_0 k_{x\varepsilon}}{k_0 \varepsilon_r} \qquad k_{x\varepsilon} = \sqrt{\varepsilon_r k_0^2 - k_z^2}$$

SPECTRAL GREEN'S FUNCTION (SGF) (2)

Unit-amplitude line sources:

$$J_i(x,z) = 1 \cdot \delta(x + h_s)\delta(z)\ \mathrm{A/m^2}$$

$$M_i(x,z) = 1 \cdot \delta(x + h_s)\delta(z)\ \mathrm{V/m^2}$$

$$\Downarrow$$

TE $$\tilde{E}_y(k_z) = -\frac{jZ_0^{TE} Z_\varepsilon^{TE} \sin[k_{x\varepsilon}(h - h_s)]}{Z_0^{TE} \cos(k_{x\varepsilon} h) + jZ_\varepsilon^{TE} \sin(k_{x\varepsilon} h)} = \tilde{G}_{TE}(k_z)$$

TM $$\tilde{H}_y(k_z) = \frac{\cos[k_{x\varepsilon}(h - h_s)]}{Z_0^{TM} \cos(k_{x\varepsilon} h) + jZ_\varepsilon^{TM} \sin(k_{x\varepsilon} h)} = \tilde{G}_{TM}(k_z)$$

SINGULARITIES OF THE SGF (TE CASE) (1)

Let us consider the SGF as a function of the complex variable k_z

$$\tilde{G}_{TE}(k_z) = -\frac{j\eta_0 k_0 \sin\left[k_{x\varepsilon}(h - h_s)\right]}{k_{x\varepsilon}\cos(k_{x\varepsilon}h) + jk_{x0}\sin(k_{x\varepsilon}h)}$$

$$k_{x0} = \sqrt{k_0^2 - k_z^2}$$

$$k_{x\varepsilon} = \sqrt{\varepsilon_r k_0^2 - k_z^2}$$

1) POLES

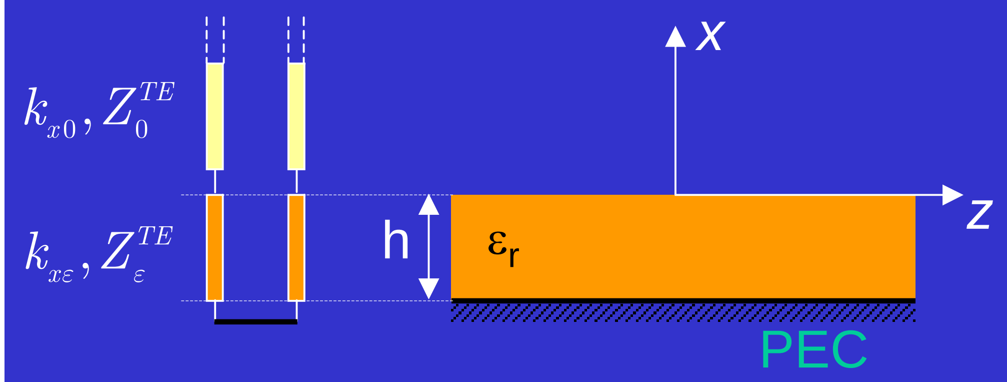
They are the zeros of the denominator of the SGF:

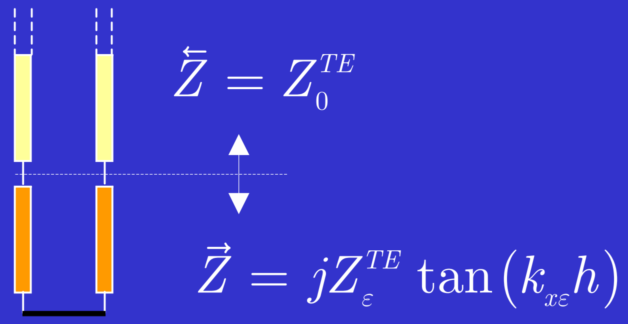
$$k_{x\varepsilon}\cos(k_{x\varepsilon}h) + jk_{x0}\sin(k_{x\varepsilon}h) = 0$$

$$\Longleftrightarrow \quad Z_0^{TE}\cos(k_{x\varepsilon}h) + jZ_\varepsilon^{TE}\sin(k_{x\varepsilon}h) = 0$$

$$\Longleftrightarrow \quad \boxed{Z_0^{TE} + jZ_\varepsilon^{TE}\tan(k_{x\varepsilon}h) = 0}$$

TRANSVERSE RESONANCE





$$\overleftarrow{Z} + \overrightarrow{Z} = 0$$

$$\boxed{Z_0^{TE} + jZ_\varepsilon^{TE} \tan\left(k_{x\varepsilon} h\right) = 0}$$

The poles of the SGF are the zeros of the characteristic equation, i.e., they are the propagation constants of the modes supported by the GDS

SINGULARITIES OF THE SGF (TE CASE) (2)

2) BRANCH POINTS

They originate from the presence of the two-valued square-root functions

$$\tilde{G}_{TE}(k_z) = -\frac{j\eta_0 k_0 \sin[k_{x\varepsilon}(h - h_s)]}{k_{x\varepsilon}\cos(k_{x\varepsilon}h) + jk_{x0}\sin(k_{x\varepsilon}h)}$$

$$k_{x0} = \pm\sqrt{k_0^2 - k_z^2}$$

$$k_{x\varepsilon} = \pm\sqrt{\varepsilon_r k_0^2 - k_z^2}$$

The SGF is an even function of $k_{x\varepsilon}$ ➡ No branch point arises from $k_{x\varepsilon}$

Two branch points at $k_z = \pm k_0$ arise from k_{x0}

SOMMERFELD BRANCH CUTS

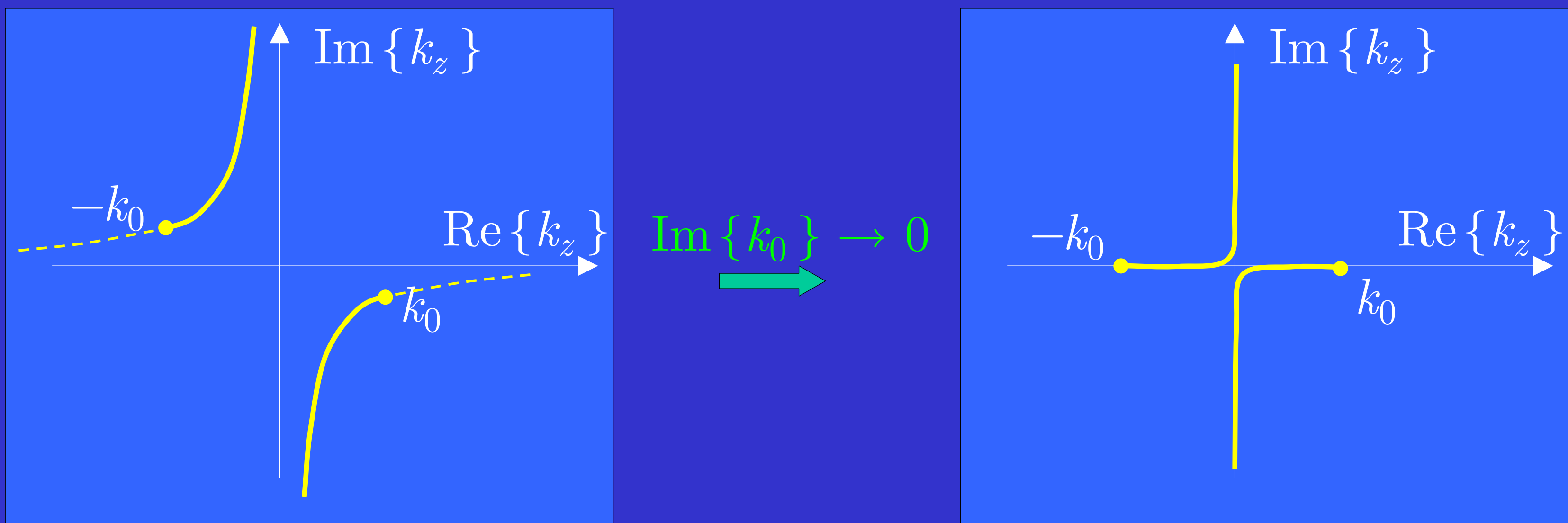
Branch cuts may be defined by arbitrarily joining the branch points $k_z = \pm k_0$ and the point at the complex infinity

By enforcing the finiteness of $e^{-jk_{x0}x}$ for $x \to +\infty$

$$\mathrm{Im}\{k_{x0}\} \leq 0$$

The Sommerfeld branch cuts are thus defined by the condition

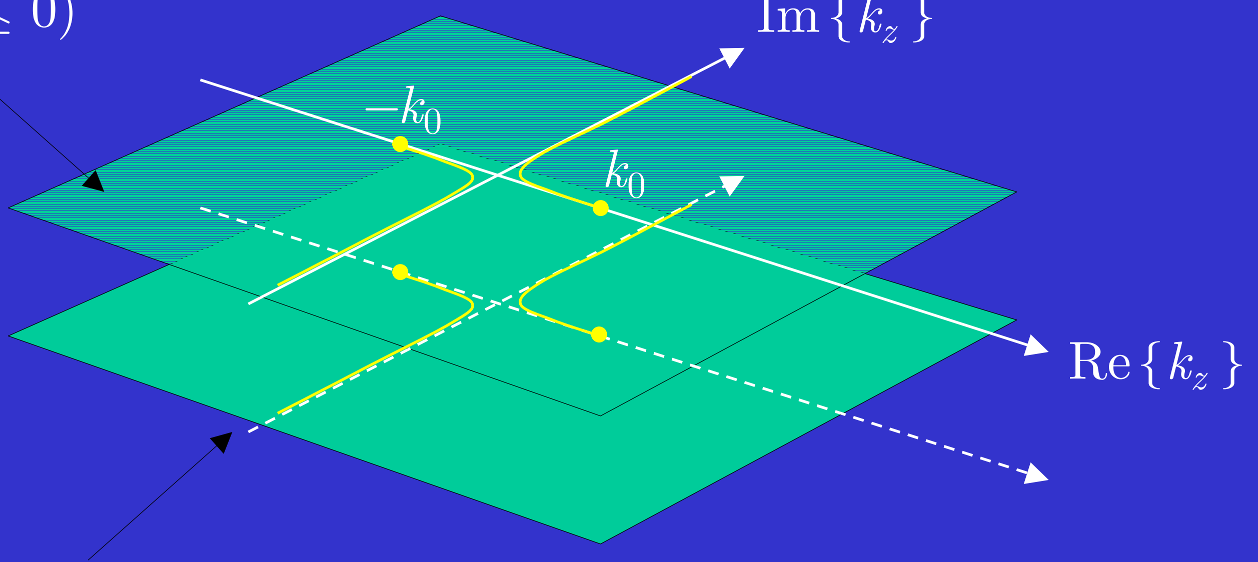
$$\mathrm{Im}\{k_{x0}\} = 0$$



RIEMANN SURFACE OF THE SGF

TOP (or proper) sheet

$(\mathrm{Im}\{k_{x0}\} \leq 0)$



BOTTOM (or improper) sheet

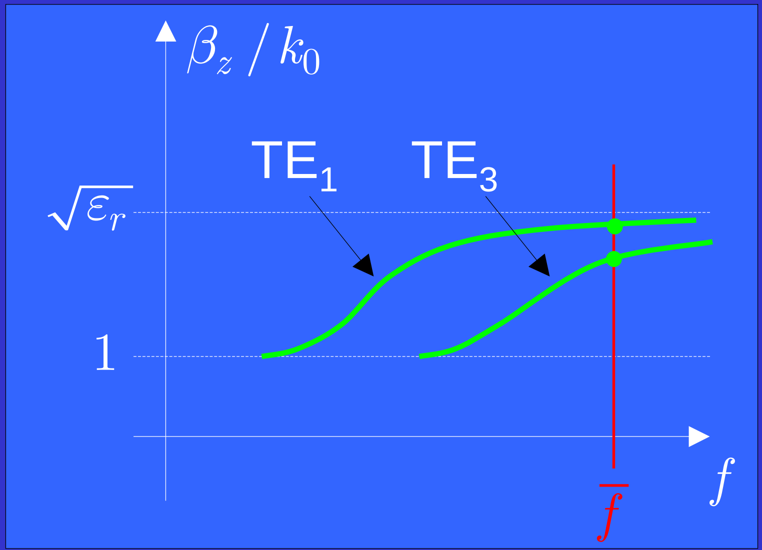
$(\mathrm{Im}\{k_{x0}\} \geq 0)$

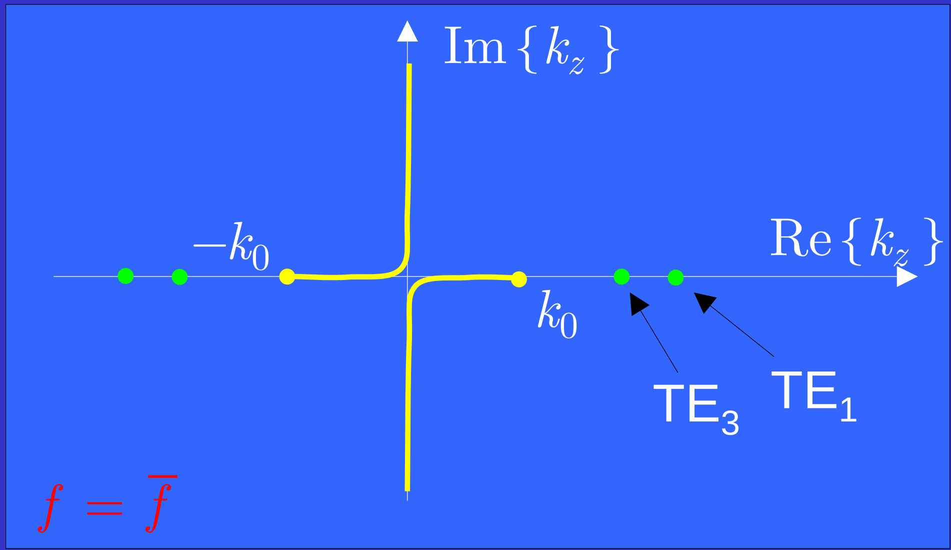
PROPER POLES OF THE SGF

Proper poles correspond to bound modes (surface waves) of the GDS

Proper eigenfunction of a lossless GDS have only real eigenvalues:

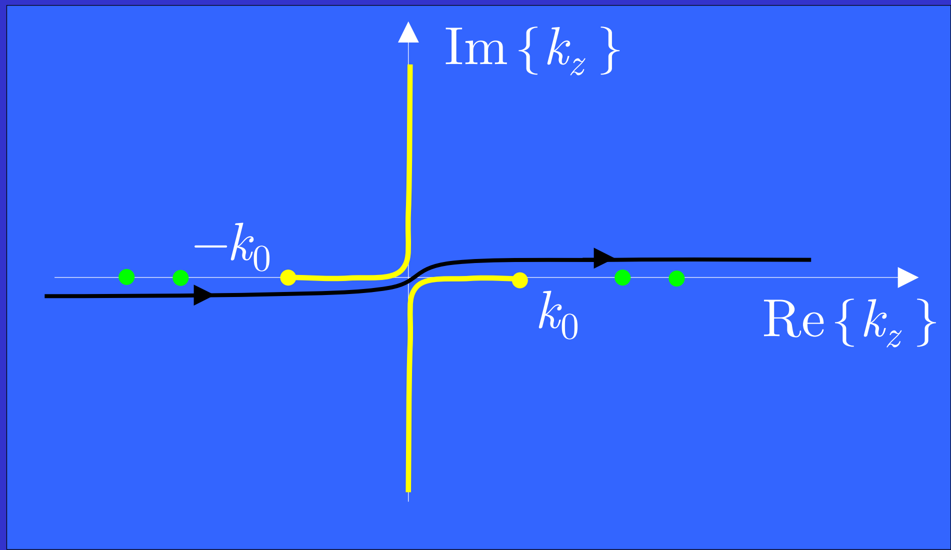
$$k_z = \beta_z \in \Re$$





FIELD AT THE AIR-SLAB INTERFACE (1)

Inverse Fourier transform: integration path in the complex plane

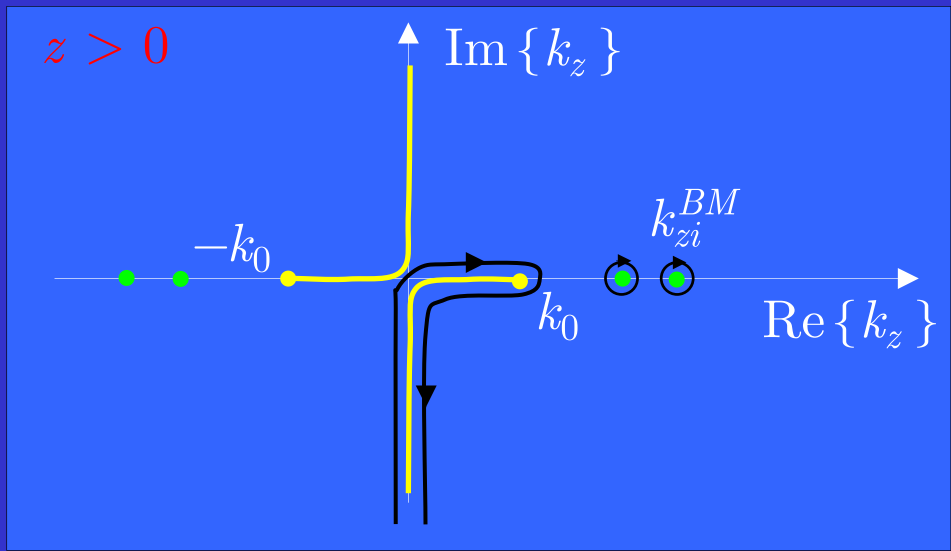


$$E_y(z) = \frac{1}{2\pi} \int_{\Re} \tilde{E}_y(k_z) e^{-jk_z z} \mathrm{d}k_z$$

FIELD AT THE AIR-SLAB INTERFACE (2)

Path deformation in the lower half plane (z > 0) of the TOP Riemann sheet:

SPECTRAL REPRESENTATION

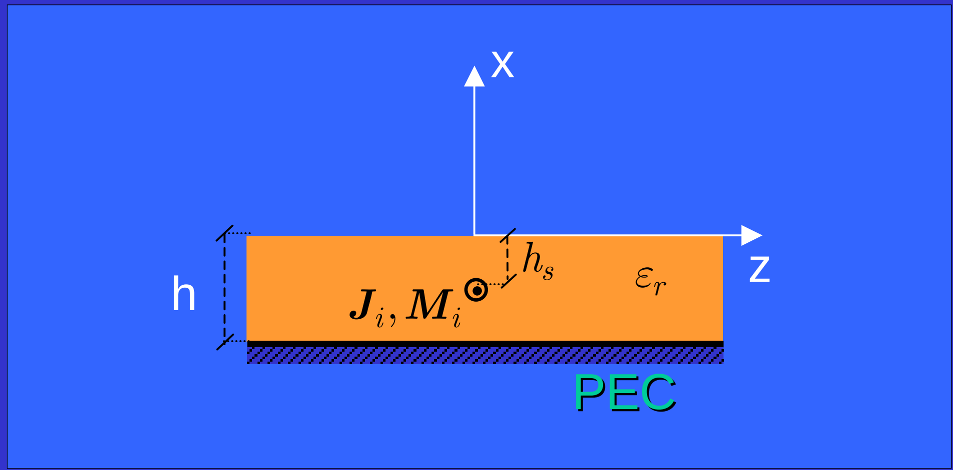


$$E_y(z) = \underbrace{\frac{1}{2\pi}\int_{\text{Branch Cut}} \tilde{E}_y(k_z)\, e^{-jk_z z}\, \mathrm{d}k_z}_{\text{Continuous spectrum}} - \underbrace{j\sum_i \mathrm{Res}\left[\tilde{E}_y\left(k_{zi}^{BM}\right)\right] e^{-jk_{zi}^{BM} z}}_{\text{Discrete spectrum (surface waves)}}$$

STEEPEST-DESCENT-PLANE

REPRESENTATION

FIELD IN THE UPPER HALF SPACE



Propagation of each spectral component:

$$E_y(x,z) = \frac{1}{2\pi} \int_{\Re} \tilde{E}_y(k_z) e^{-jk_{x0}x} e^{-jk_z z} \mathrm{d}k_z$$

$$k_{x0} = \sqrt{k_0^2 - k_z^2}, \quad \mathrm{Im}\{k_{x0}\} \leq 0$$

In order to enforce the radiation condition at infinity

CHANGE OF VARIABLES

Polar coordinates in both the spatial and spectral domains:

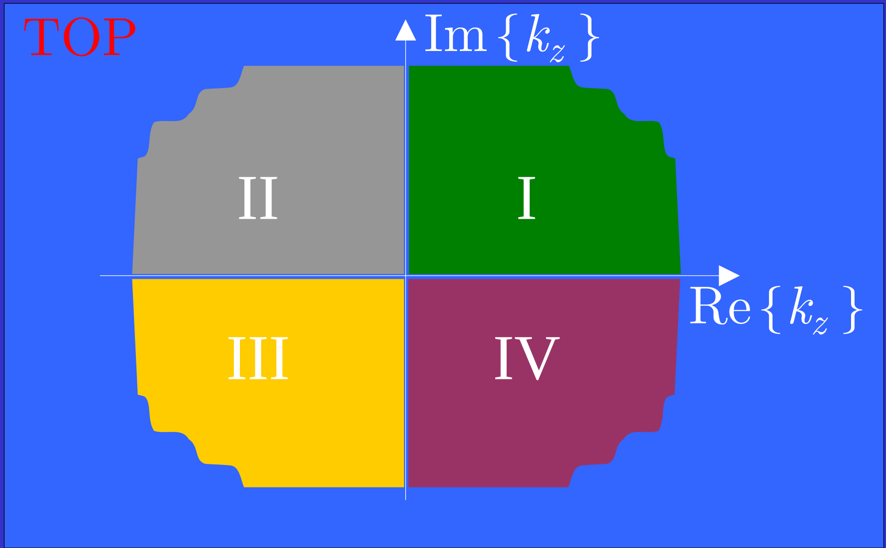
$$\boxed{\begin{aligned} x &= \rho\cos\theta & \qquad k_{x0} &= k_0\cos\phi \\ z &= \rho\sin\theta & \qquad k_z &= k_0\sin\phi \end{aligned}}$$

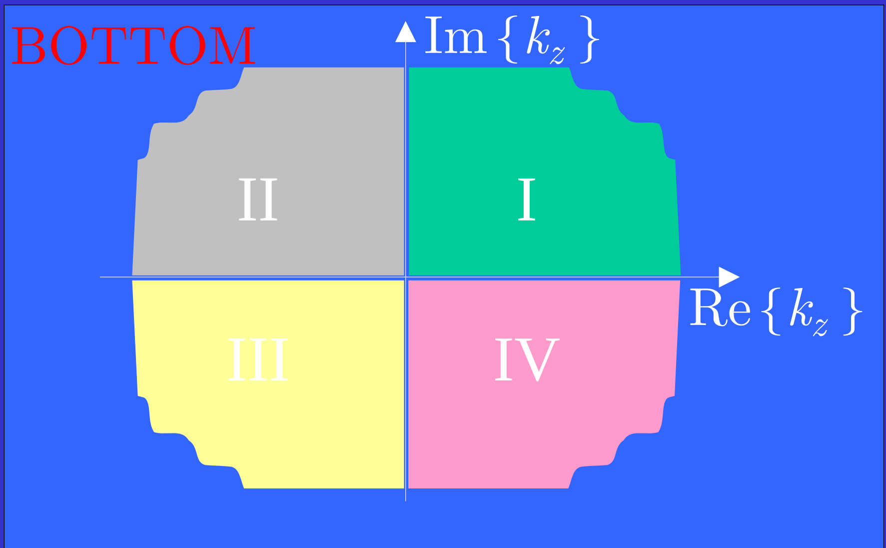
$$\Rightarrow \quad E_y(x,z) = \frac{1}{2\pi}\int_{\Re} \tilde{E}_y(k_z)\, e^{-jk_{x0}x}e^{-jk_z z}\,\mathrm{d}k_z$$

$$E_y(\rho,\theta) = \frac{1}{2\pi}\int_{C} \tilde{E}_y(k_0\sin\phi)\, e^{-jk_0\rho\cos(\theta-\phi)}\, k_0\cos\phi\;\mathrm{d}\phi =$$

Integration path in the complex ϕ plane

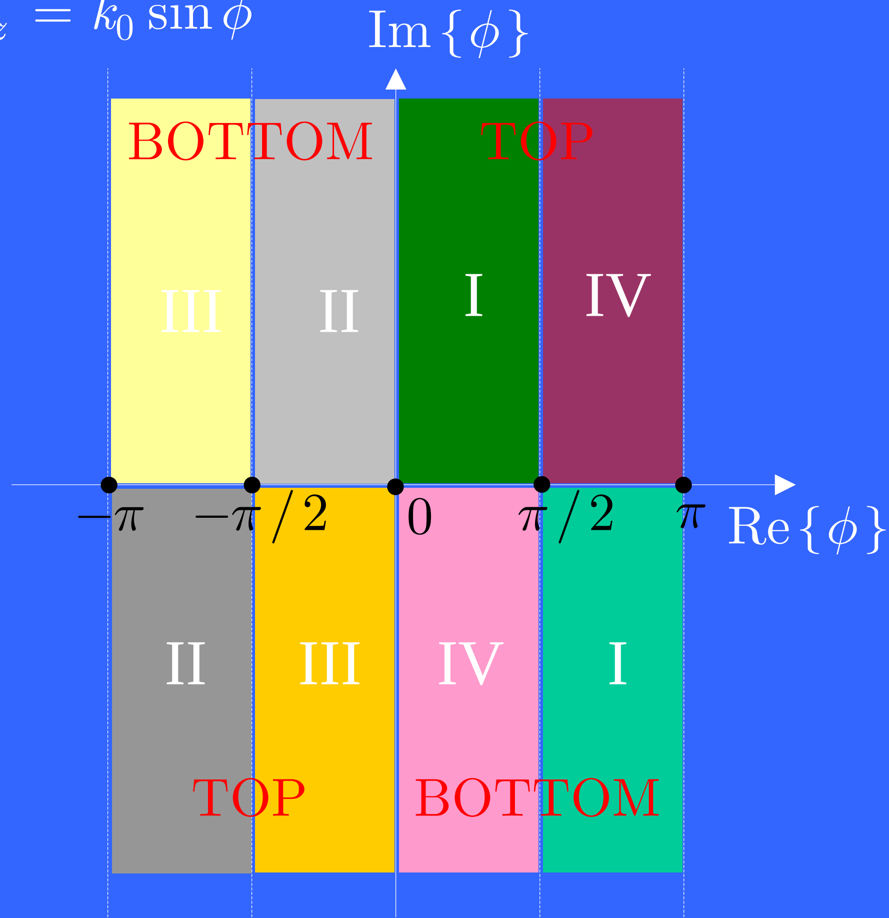
THE STEEPEST-DESCENT PLANE (1)



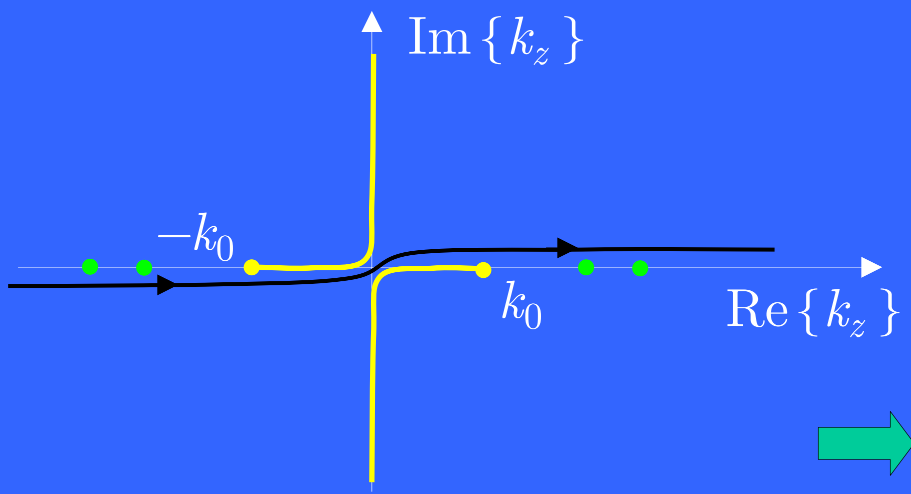


$$k_{x0} = k_0 \cos\phi$$

$$k_z = k_0 \sin\phi$$

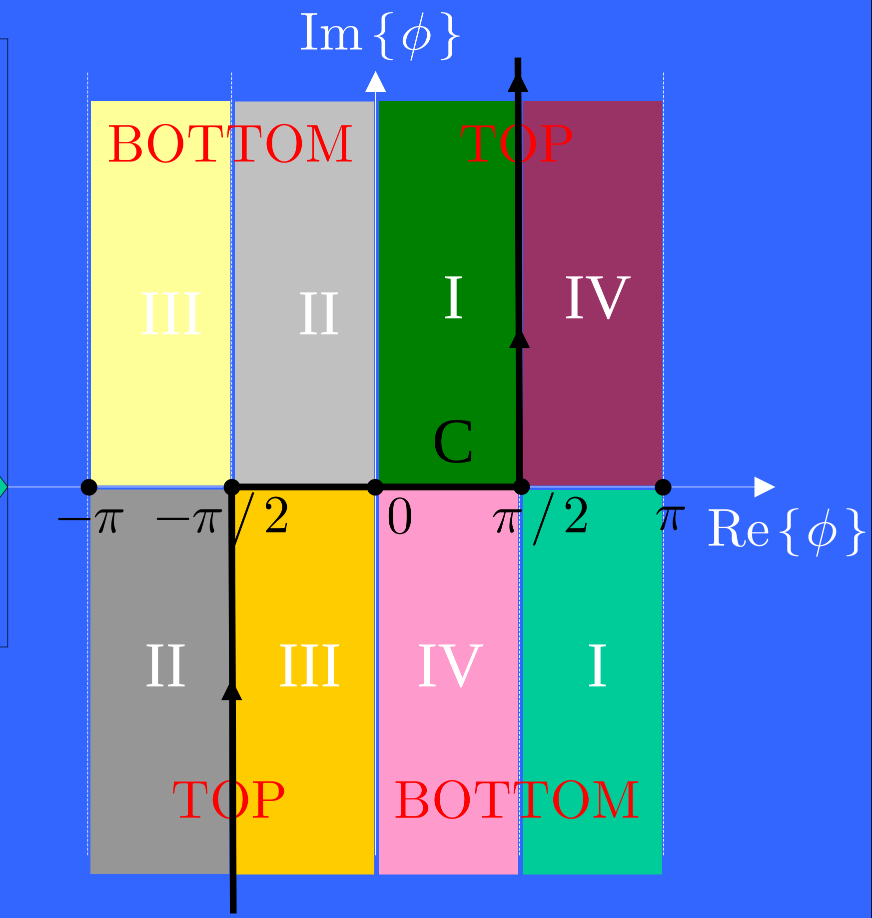


THE STEEPEST-DESCENT PLANE (2)

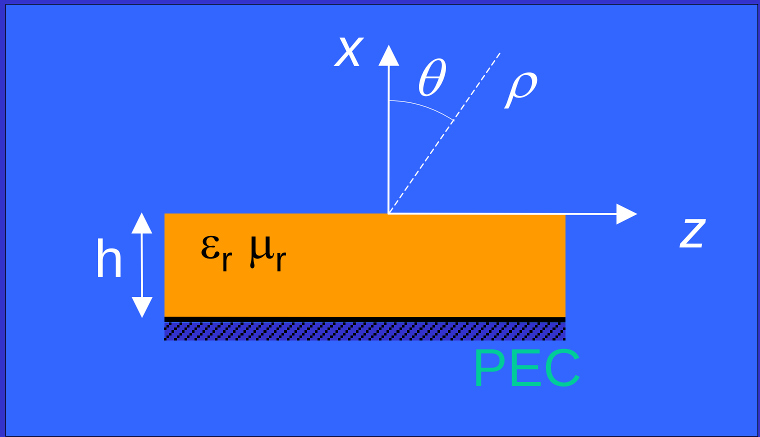


Integration along the real axis

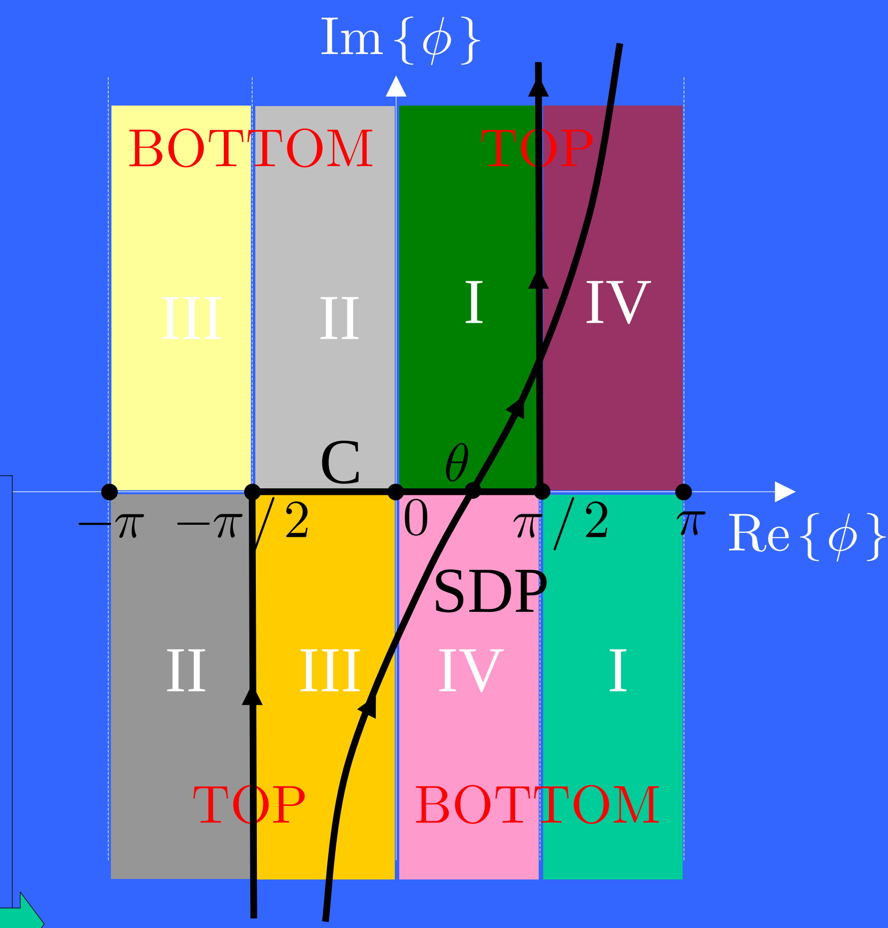
The corresponding integration path C

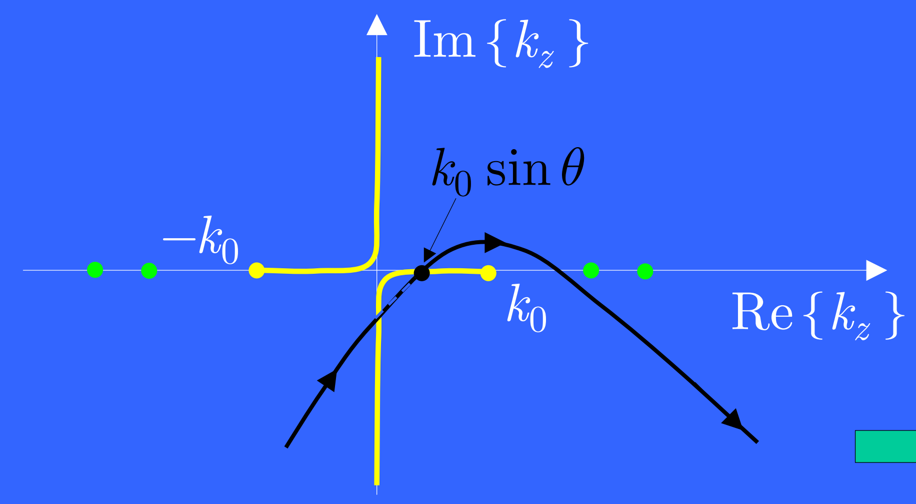


THE STEEPEST-DESCENT PLANE (3)



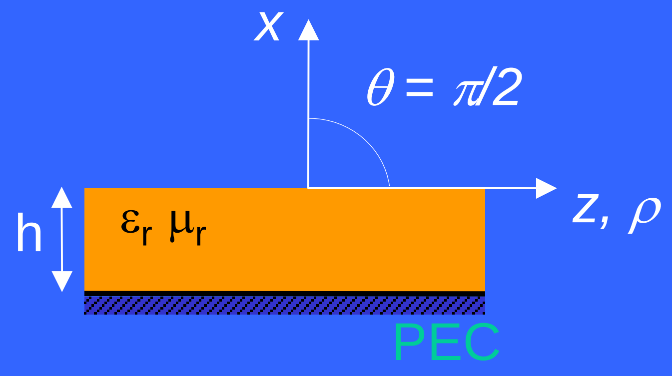
The steepest-descent path



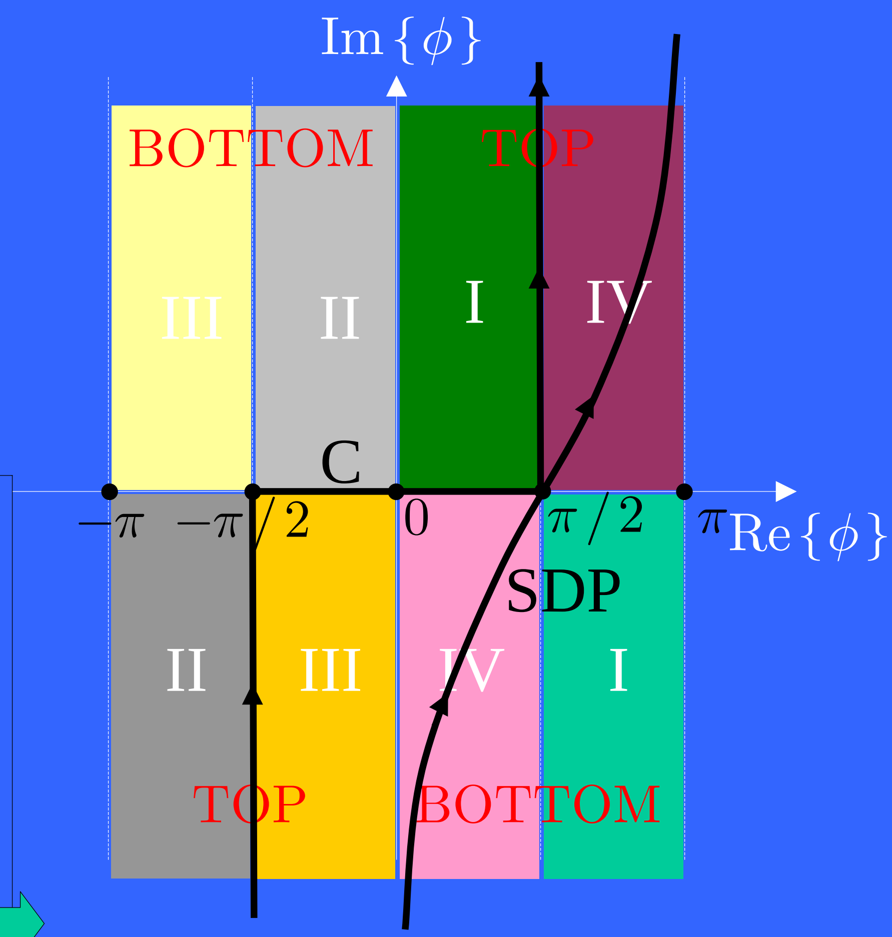


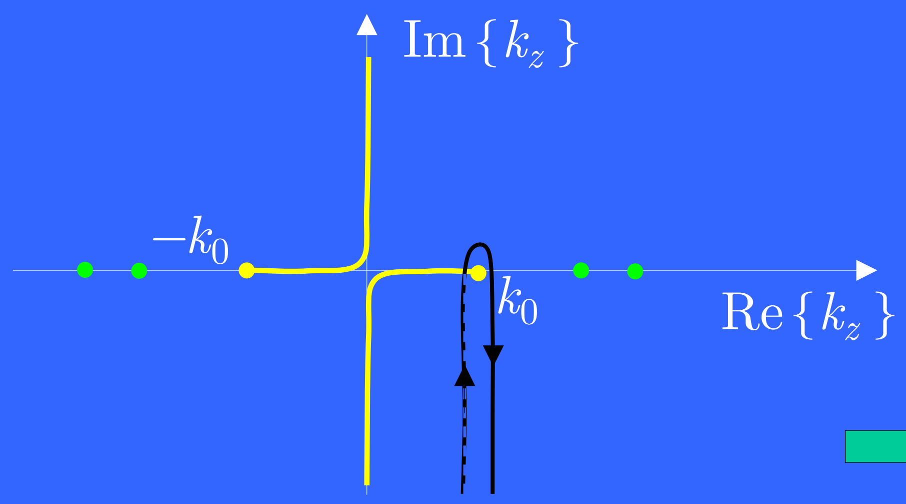
Integration along the SDP

THE STEEPEST-DESCENT PLANE (4)



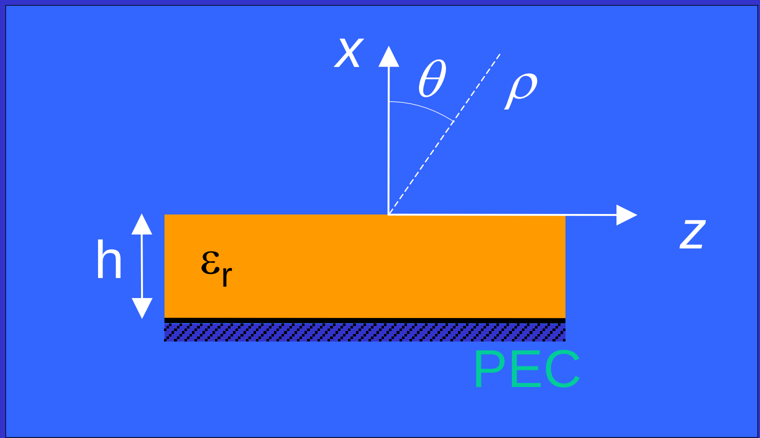
The steepest-descent path





Integration along the SDP

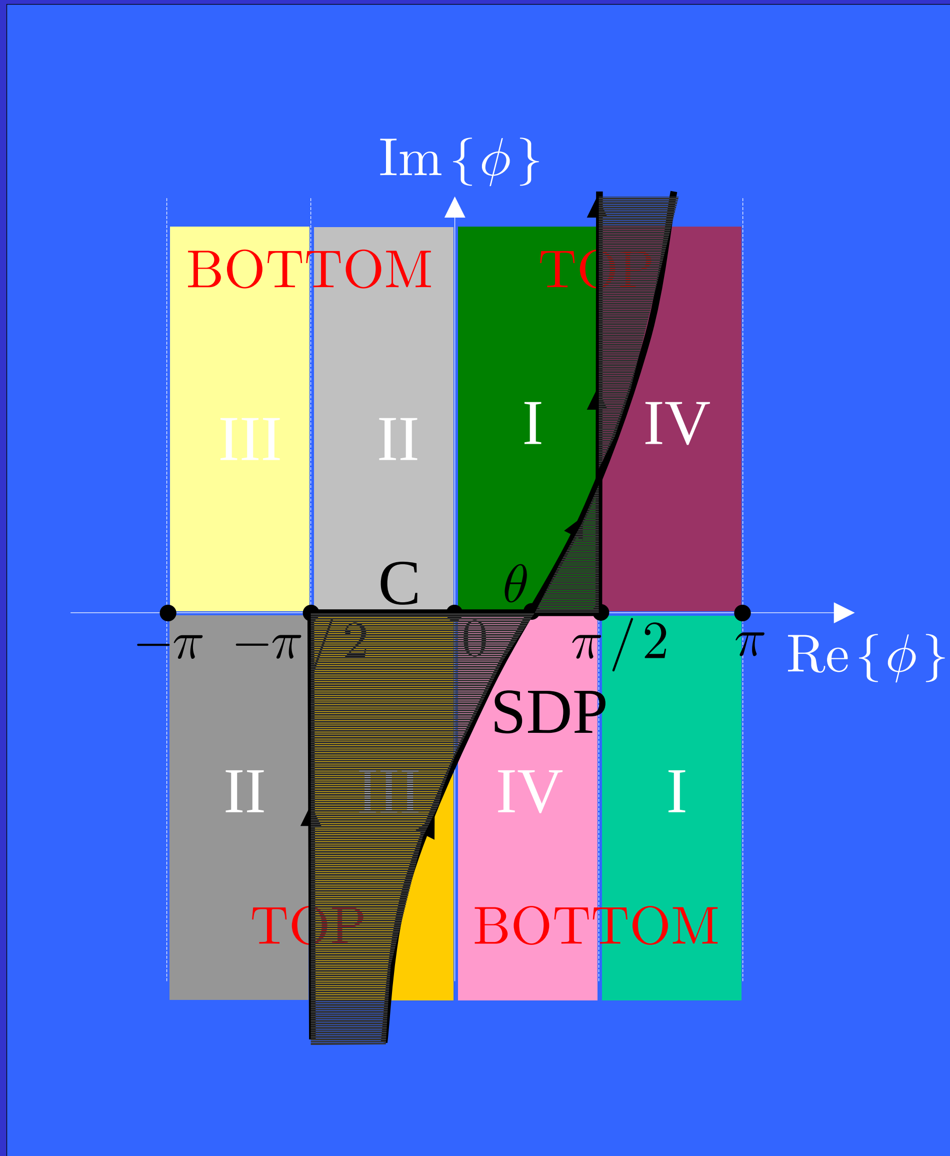
THE SDP REPRESENTATION (1)



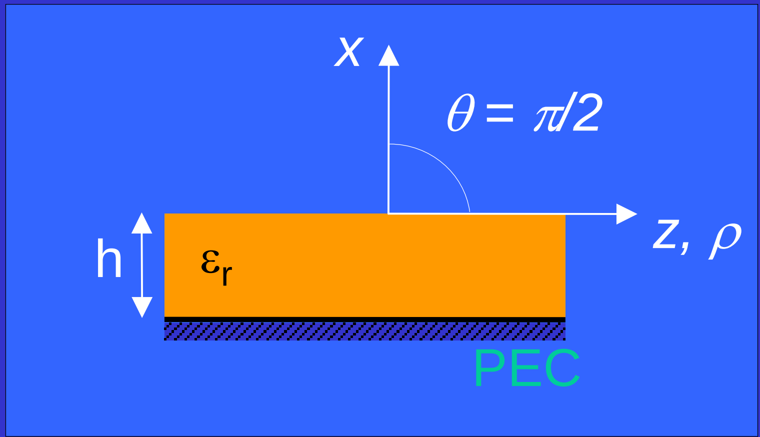
$$E_y(\rho,\theta) = k_0 F(\theta)\frac{e^{-j(k_0\rho-\pi/4)}}{\sqrt{2\pi k_0\rho}} +$$

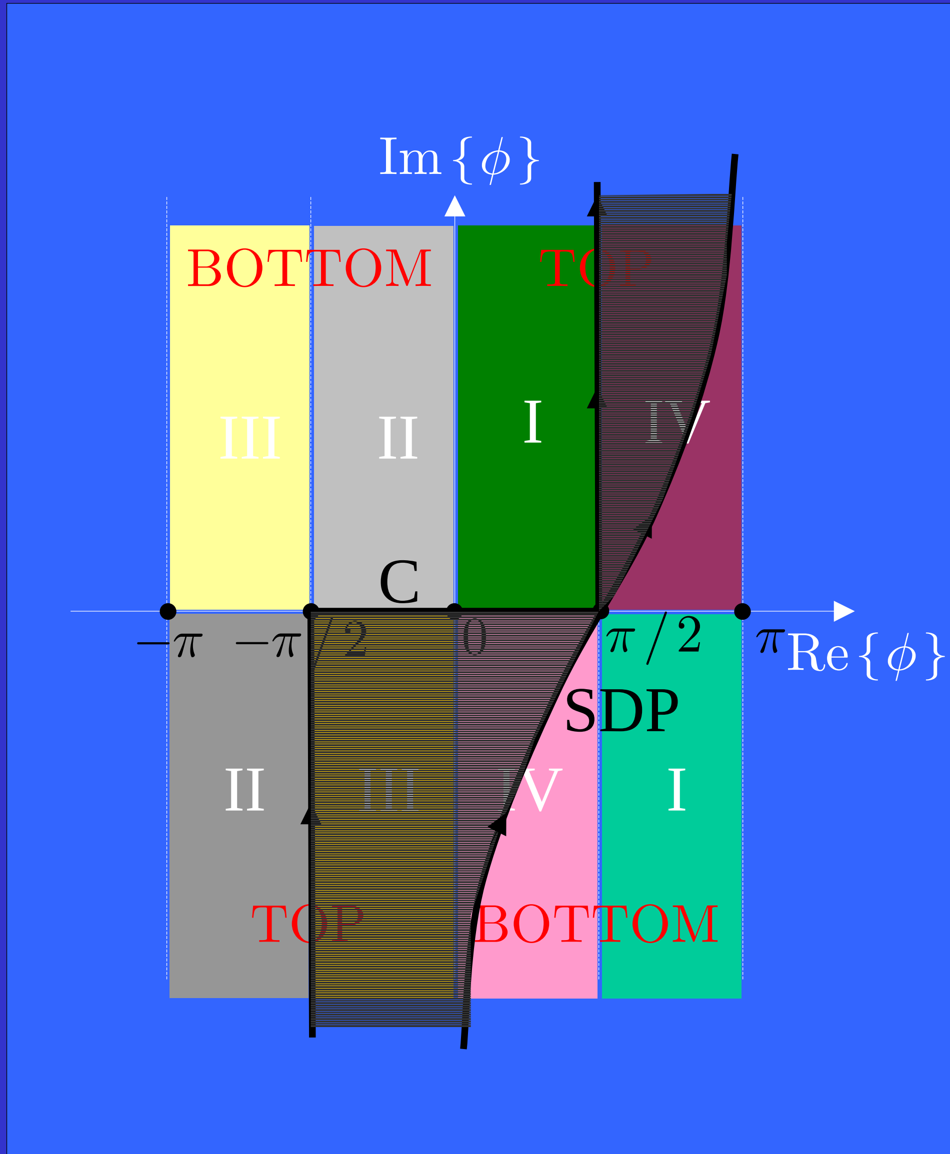
+Residue contributions

from captured poles



THE SDP REPRESENTATION (2)





$$E_y(\rho,\theta) = \frac{k_0}{2} F''\left(\frac{\pi}{2}\right) \frac{e^{-j(k_0\rho - 3\pi/4)}}{\sqrt{2\pi(k_0\rho)^3}} +$$

+Residue contributions

from captured poles

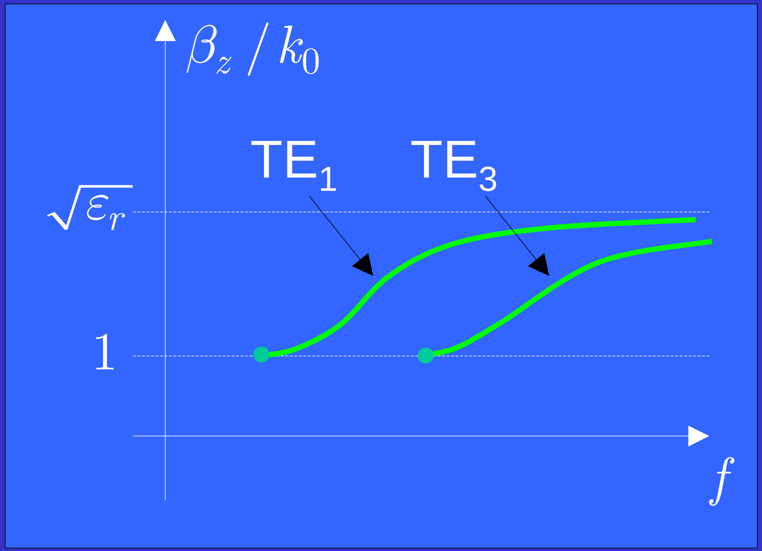
LEAKY MODES

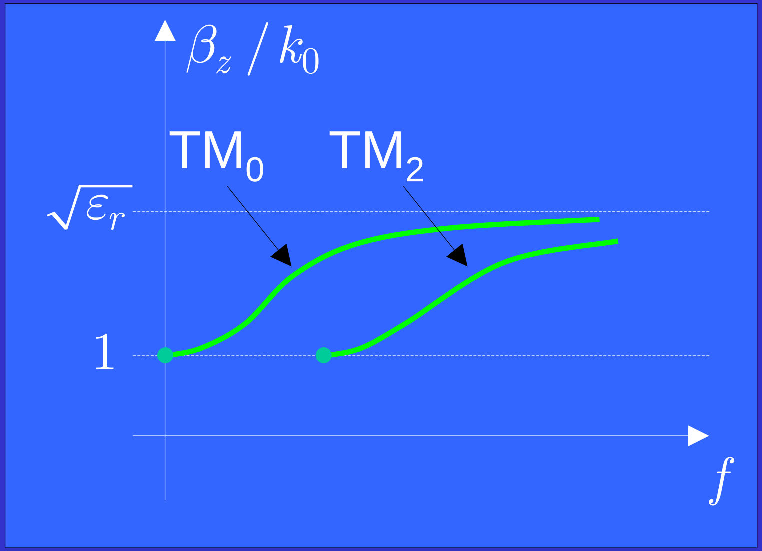
AND FAR-FIELD PATTERNS

IMPROPER POLES OF THE SGF (1)

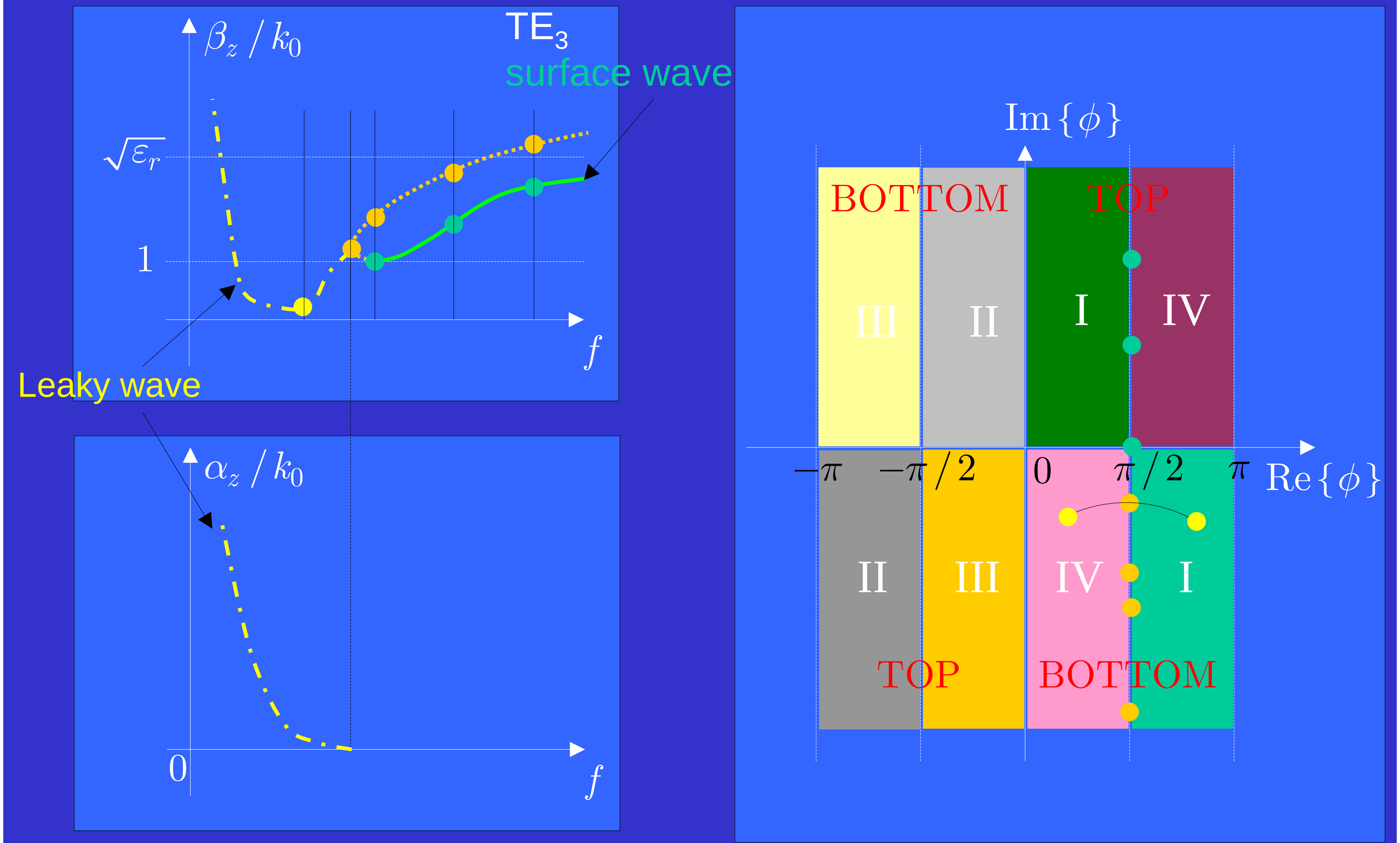
Improper poles arise, e.g., as the analytic continuation of proper poles below the cutoff frequency of the corresponding surface wave

TE and TM surface waves and their cutoffs:



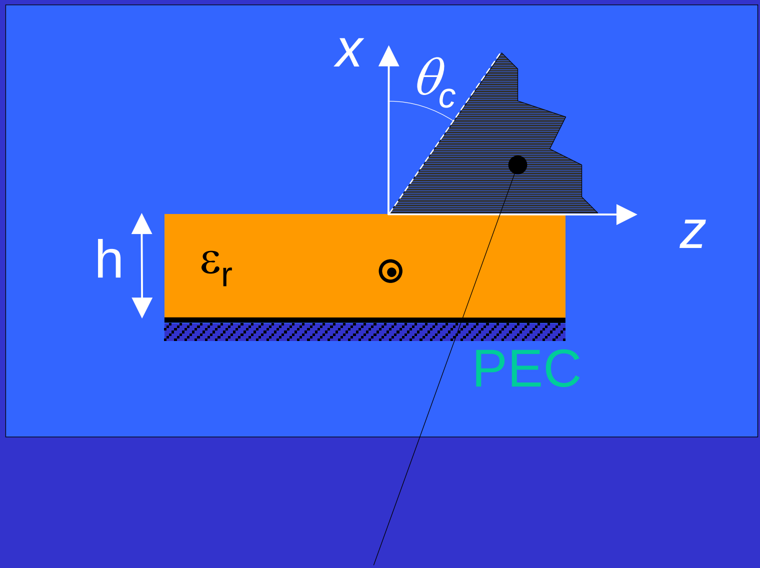


IMPROPER POLES OF THE SGF (2)



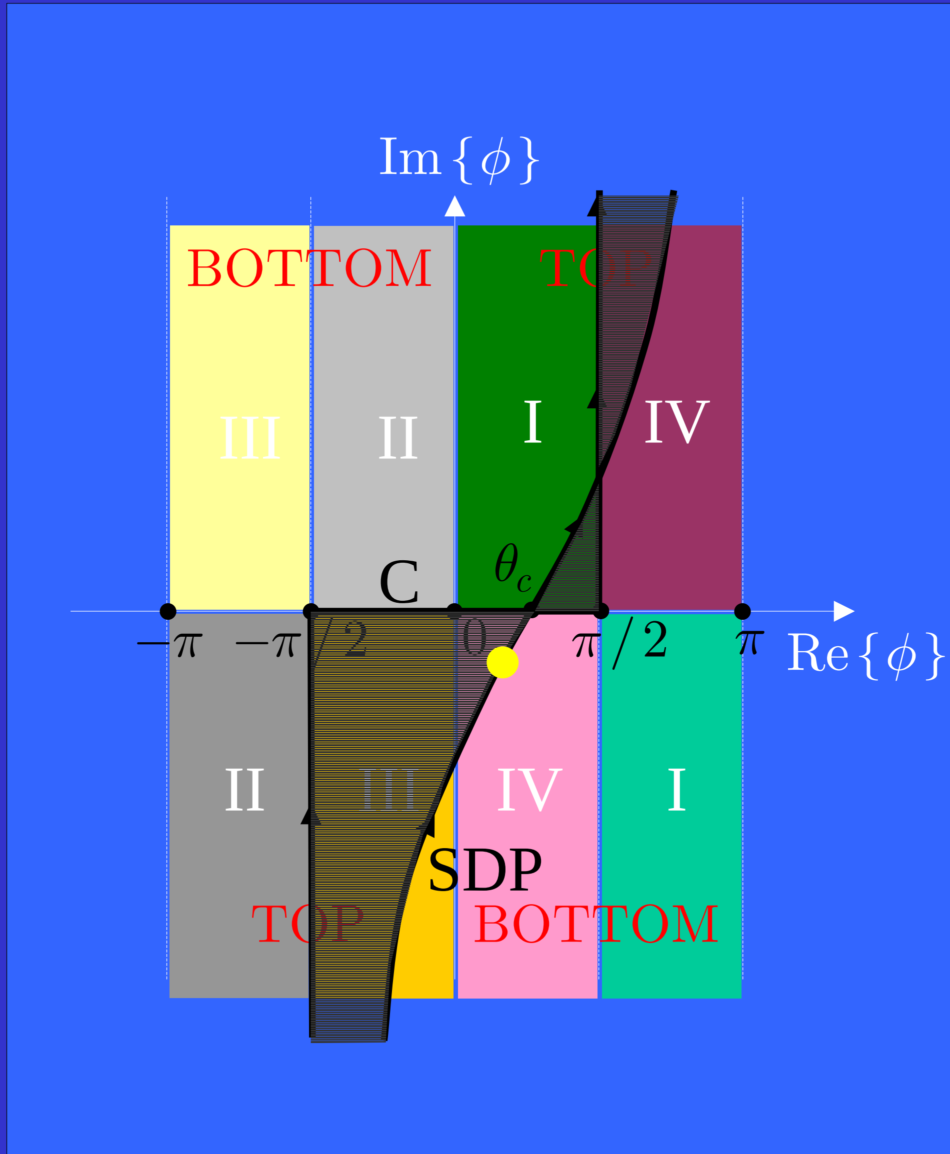
LEAKY MODES: ANGLE OF DEFINITION

Angle of definition (or capture) θ_c



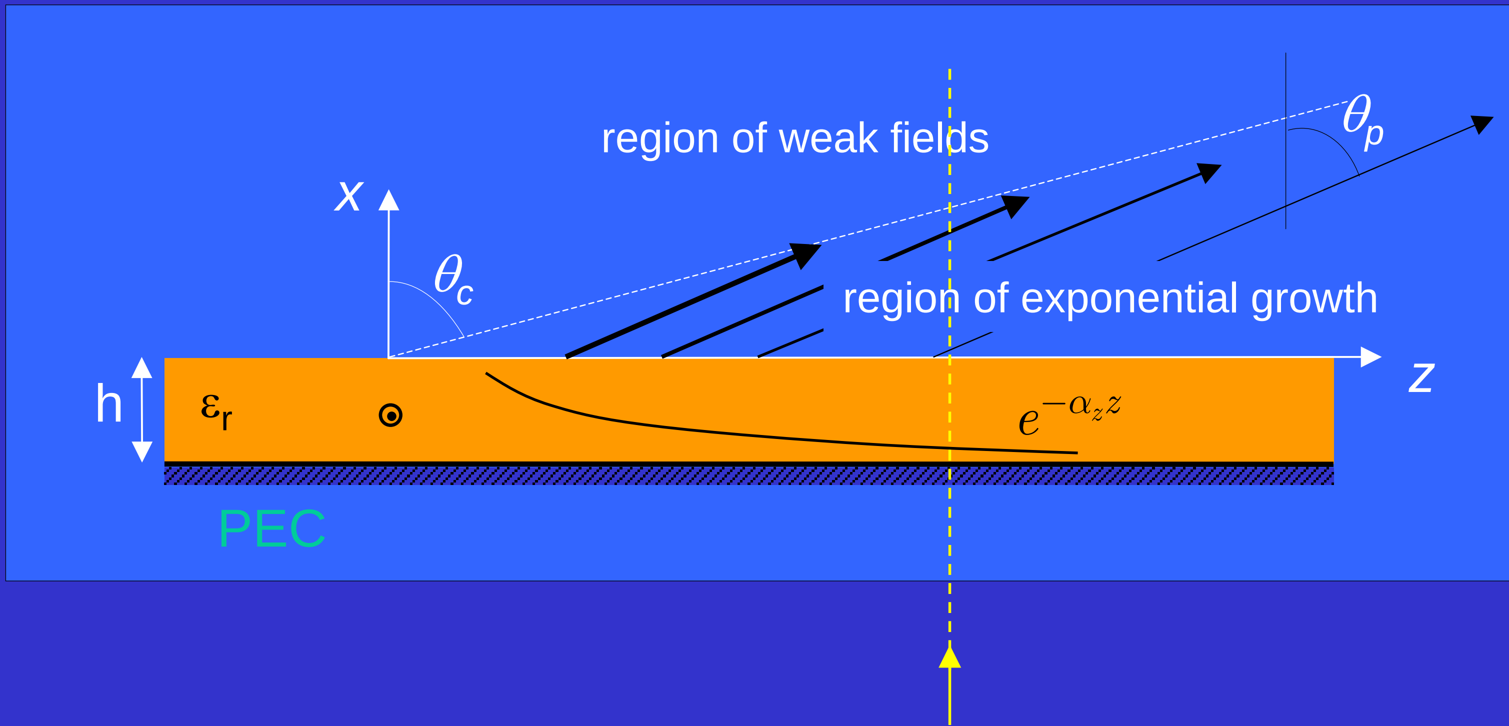
Angular region where the leaky mode contributes to the field in the SDP representation:

$$\theta > \theta_c$$



LEAKY MODES: POWER FLOW

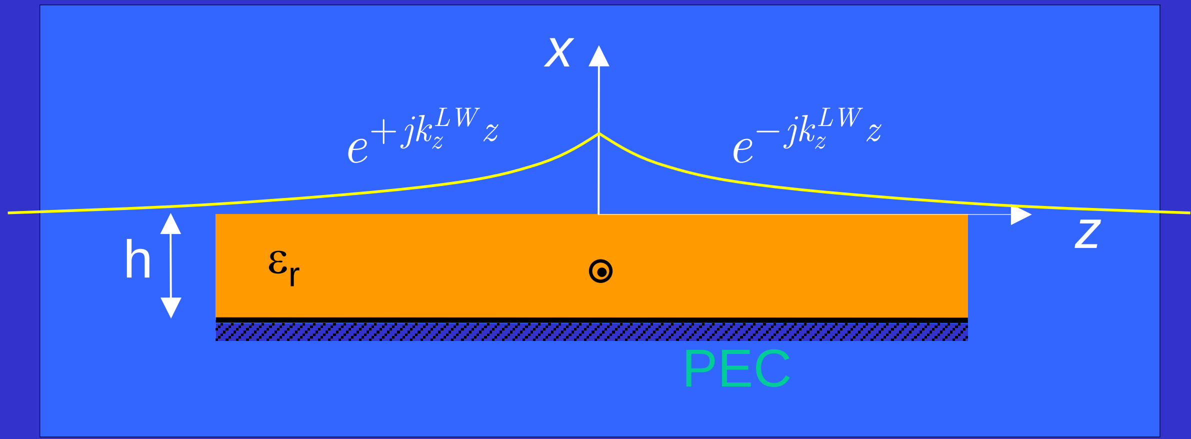
$$\beta_z = k_0 \sin\theta_p$$



Within the angle of definition, the leaky-mode field transversely increases exponentially

For small values of the attenuation constant α_z : $\theta_p \simeq \theta_c$

LEAKY MODES: APERTURE FIELD



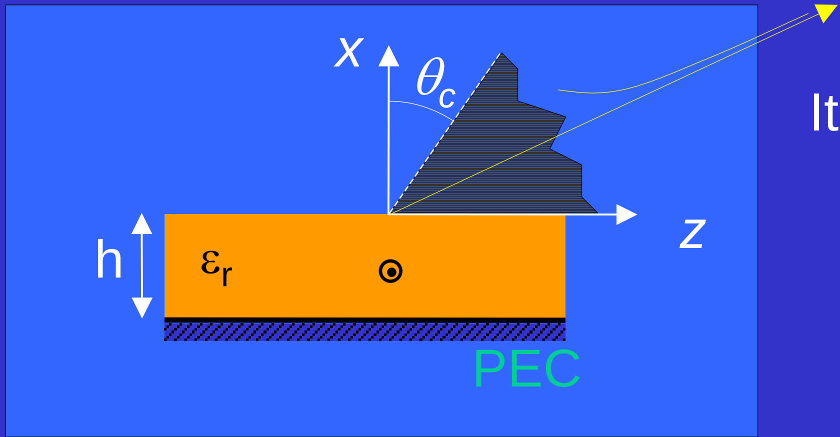
$$E_y^{LW}(z) = -j\mathrm{Res}\left[\tilde{E}_y\left(k_z^{LW}\right)\right]e^{-jk_z^{LW}|z|}$$

The corresponding far-field radiated by this aperture distribution is simply obtained through a Fourier transform:

$$E_{ff}^{LW}(\theta) \propto \cos\theta \tilde{E}_y^{LW}(k_0 \sin\theta) = \frac{2jk_z^{LW}\cos\theta}{(k_0\sin\theta)^2 - \left(k_z^{LW}\right)^2}$$

LEAKY MODES AND RADIATION PATTERNS

The leaky mode radially decreases exponentially in all the directions within its angle of definition



⇓

It does not contribute directly to the far field

HOWEVER:

The leaky mode may dominate the aperture field at the air-slab interface

⇓

The radiation pattern may be well approximated by means of the Fourier transform of the leaky-mode aperture field alone

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