# High-frequency Incremental Techniques for Scattering and Diffraction

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# SUMMARY

- **Background**
  - ✓ Basic terminology
  - ✓ GTD versus PTD
  - ✓ Basic concepts of Incremental Theories

- **Overwiev of Incremental Theories**
  - ✓ Current based methods
  - ✓ Field based methods

- **Incremental Theory of Diffraction (ITD)**
  - ✓ Localization process
  - ✓ Wedge-shaped configurations
  - ✓ Cylindrical configuration
  - ✓ Double edge diffraction

# HF SCATTERING FROM LARGE OBJECTS

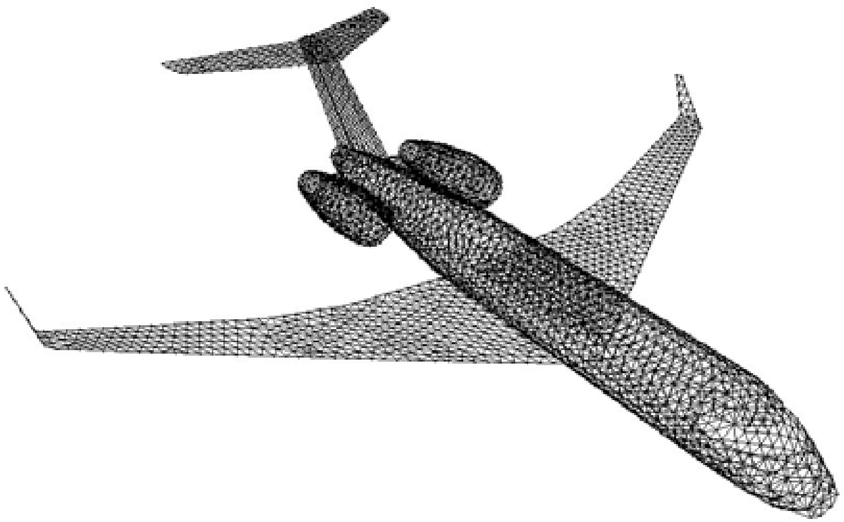
- An efficient and accurate description of scattering phenomena at high frequency is of interests in many applications.

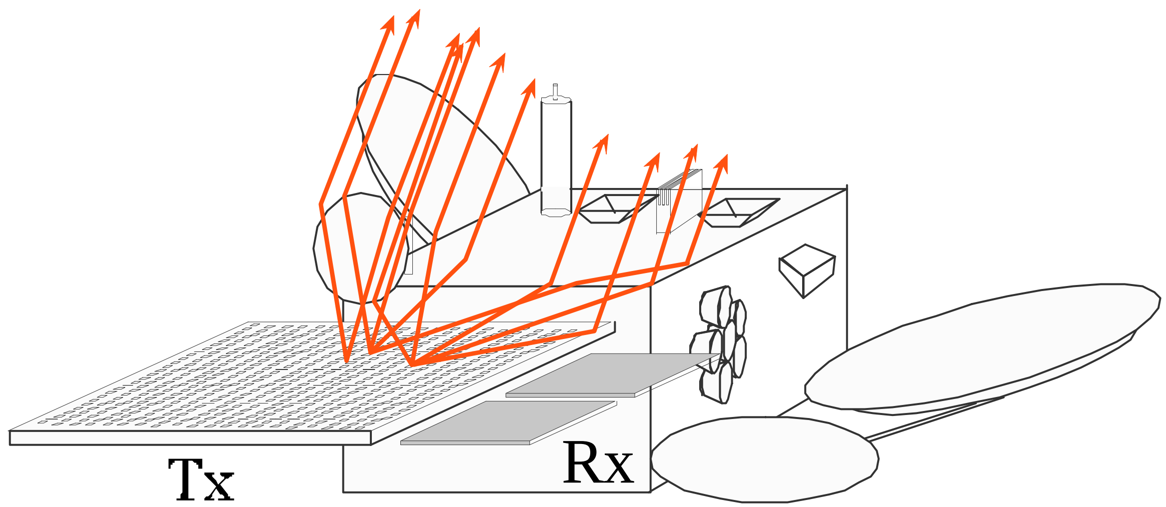
→ design and analysis of antennas (e.g., reflectors)

→ prediction of Radar Cross Section

→ propagation in complex environments (e.g., urban propagation)

→ antenna installation and inter-antenna coupling (e.g., on ships, aircrafts, space platforms…)





# HF EDGE DIFFRACTION TECHNIQUES

- The most important theory that has dominated this scenario in past few decades is:

  - **Geometrical Theory of Diffraction (GTD) in its Uniform version (UTD, UAT)**

- Other important theory is:

  - **Physical Theory of Diffraction (PTD*) in its original version**

- Both UTD and PTD are ray-optical theories: at high frequency, the scattering from a complex object may be obtained as a superposition of rays emanating from "isolated" flash points.

* P. Ya Ufimtsev "Method of edge waves in the physical theory of diffraction" Soviet Radio Publication House, 1962. English Version: Air Force Foreign Technology Division, FTD-HC-23-259-71, Sept. 1971 (normal incidence).

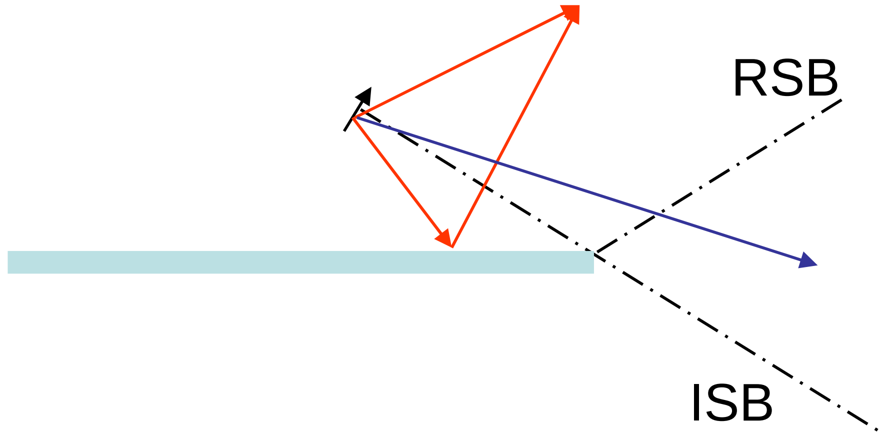
# GEOMETRICAL THEORY OF DIFFRACTION

- The *exact* field is the sum of a geometrical optics field (GO) and a diffracted field.

$$\mathbf{E} = \mathbf{E}^{go} + \mathbf{E}^{diff}$$

- The GO field accounts for *direct*, and *reflected* ray-fields.

- The space surrounding the object is divided into regions separated by the *reflection and shadow boundaries*.




- The diffracted field is the contribution which must be added to the GO to obtain the exact field.

- The GTD field provides an *asymptotic approximation* of the diffracted field.

$$\mathbf{E}^{diff} \sim \mathbf{E}^{gtd}$$

# PHYSICAL THEORY OF DIFFRACTION

- Ufimtsev's basic approach:

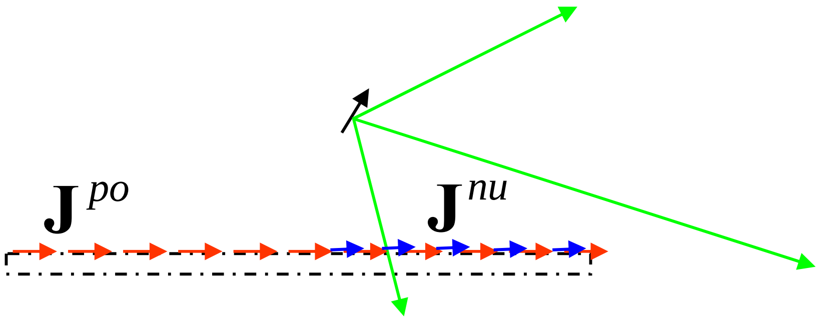
  "*First, we will investigate the rigorous solution of this problem (canonical). Then we will find its solution in the physical optics approach. The difference of these solutions determines the field created by the non uniform part of the current (fringe wave current).*"

- The *scattered* field is the sum of a physical optics field (PO) and a *fringe wave* field.

$$\boxed{\mathbf{E} = \mathbf{E}^{i} + \mathbf{E}^{po} + \mathbf{E}^{fw}}$$

- The PO field arises from the induced surface currents on the object $\mathbf{J}^{po}$.

- The FW field arises from the *non uniform* surface currents on the objects $\mathbf{J}^{nu}$.



# PHYSICAL THEORY OF DIFFRACTION

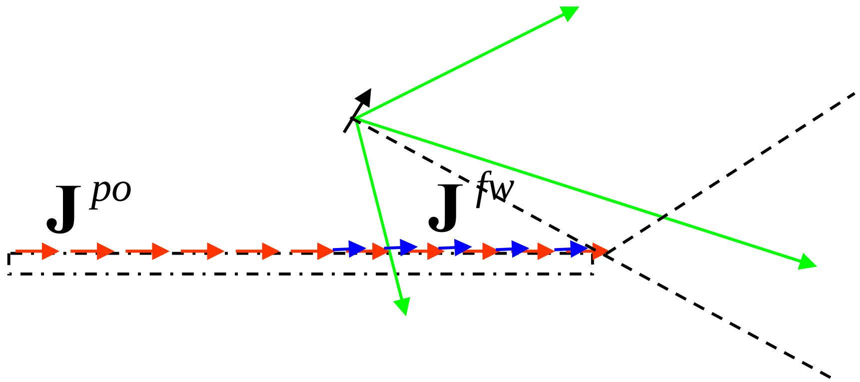
- The NU currents must be added to the PO currents to obtain the total currents, that radiates the scattered field.

$$\mathbf{J} = \mathbf{J}^{po} + \mathbf{J}^{nu}$$

- The PTD field provides an *asymptotic approximation* of the fringe wave field.

$$\mathbf{E}^{fw} \sim \mathbf{E}^{ptd}$$

- One major advantage of the PO approach is that both the incident and the PO fields are continuous across the RSB and ISB. As a consequence, also the FW field and its asymptotic approximation (PTD field) are continuous across the same SB.

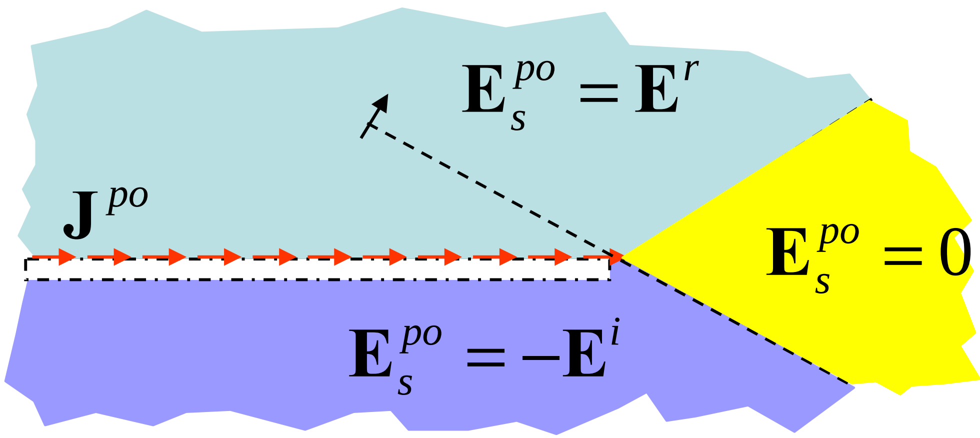


# GTD VERSUS PTD

- The PO field is divided into a PO surface contribution $\mathbf{E}_s^{po}$ and a PO diffracted field $\mathbf{E}_d^{po}$.

$$\boxed{\mathbf{E}^{po} = \mathbf{E}_s^{po} + \mathbf{E}_d^{po}}$$

- The PO surface contribution is the leading term of the asymptotic evaluation of the PO surface radiation integral for k $\rightarrow \infty$.




- As a consequence, the sum of the fringe wave and the PO diffracted field is equal to the diffracted field.

$$\boxed{\mathbf{E}_d^{po} + \mathbf{E}^{fw} = \mathbf{E}^{diff}} \quad \Rightarrow \quad \boxed{\mathbf{E}^{fw} = \mathbf{E}^{diff} - \mathbf{E}_d^{po}}$$

# GTD VERSUS PTD

- The PO diffracted field is asymptotically approximated by the PO edge-diffracted contribution $\mathbf{E}_{ed}^{po}$ .

$$\mathbf{E}_d^{po} \sim \mathbf{E}_{ed}^{po}$$

- The PO edge-diffracted contribution is the leading term of the asymptotic evaluation of the PO radiation integral for k $\rightarrow \infty$.

- The difference between the GTD field and the PO edge-diffracted field is the PTD field.

$$\mathbf{E}^{fw} = \mathbf{E}^{diff} - \mathbf{E}_d^{po} \quad \Rightarrow \quad \mathbf{E}^{ptd} = \mathbf{E}^{gtd} - \mathbf{E}_{ed}^{po}$$

- At high frequency, GO and PO predictions may be augmented by GTD and PTD field respectively.

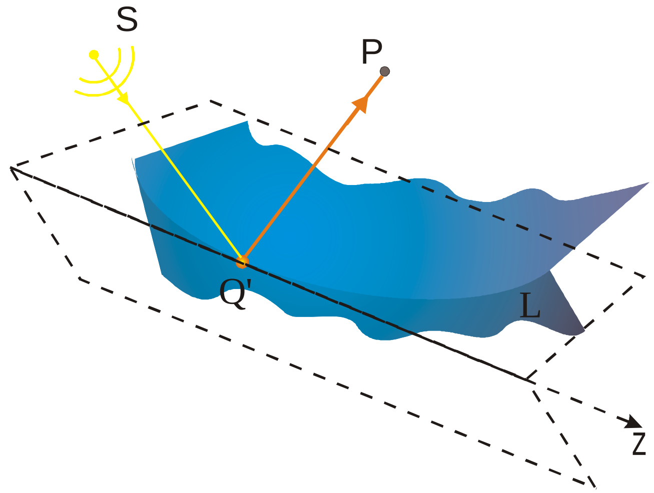
$$\mathbf{E} \sim \begin{cases} \mathbf{E}^{go} + \mathbf{E}^{gtd} \\ \mathbf{E}^{i} + \mathbf{E}^{po} + \mathbf{E}^{ptd} \end{cases}$$

# RAY METHODS AND CANONICAL PROBLEMS

- Both GTD and PTD are ray-optical methods: upon an identification of stationary surface and edge points (ray tracing), the proper ray-field is associated to each ray.

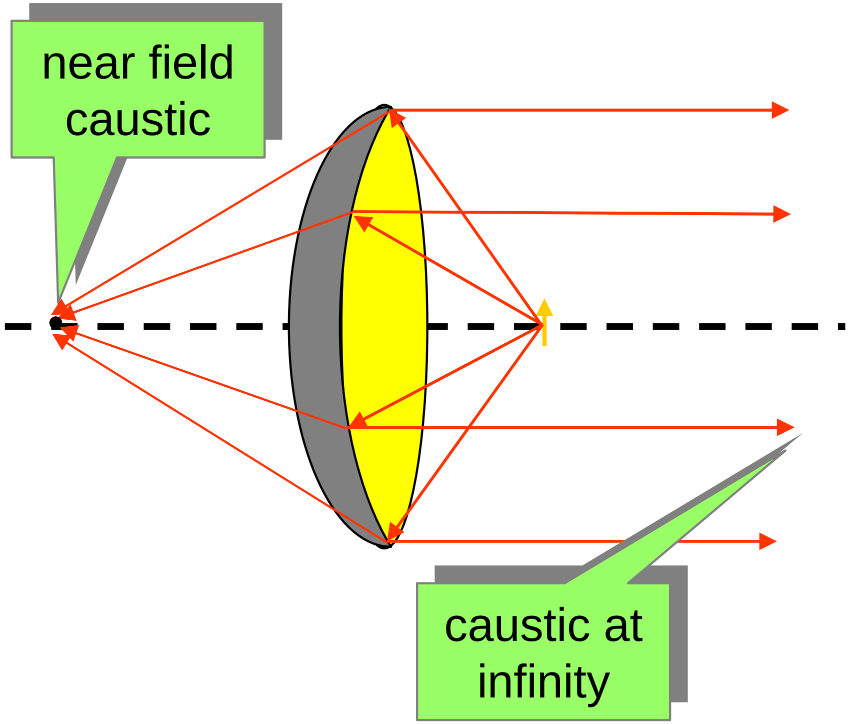
- The locality principle is essential: when a radiating object is large in terms of a wavelength the scattering and diffraction is found to be a local phenomenon,i.e., it depends only on the nature of the boundary surface and the incident field in the immediate neighborhood of the "flash" points.

- The actual scatterer is substituted by local canonical configurations at the points of reflection and diffraction.

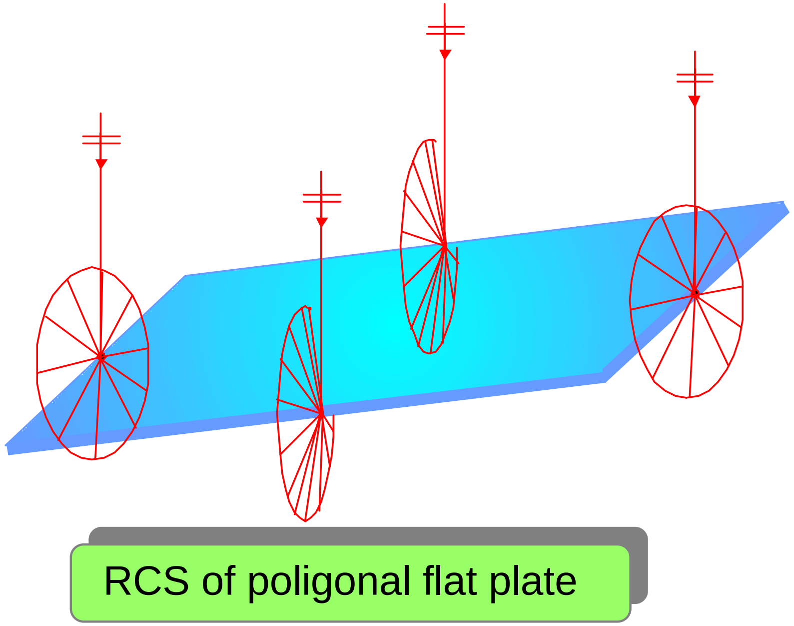


# LIMITATIONS

- Ray-Optical methods are not applicable when isolated stationary edge points cannot be identifies.




- Observation point is located at a caustic: all points along the edge, or on a finite portion, are stationary.




- Observation points is located outside the cones of diffraction: no edge stationary points exists.

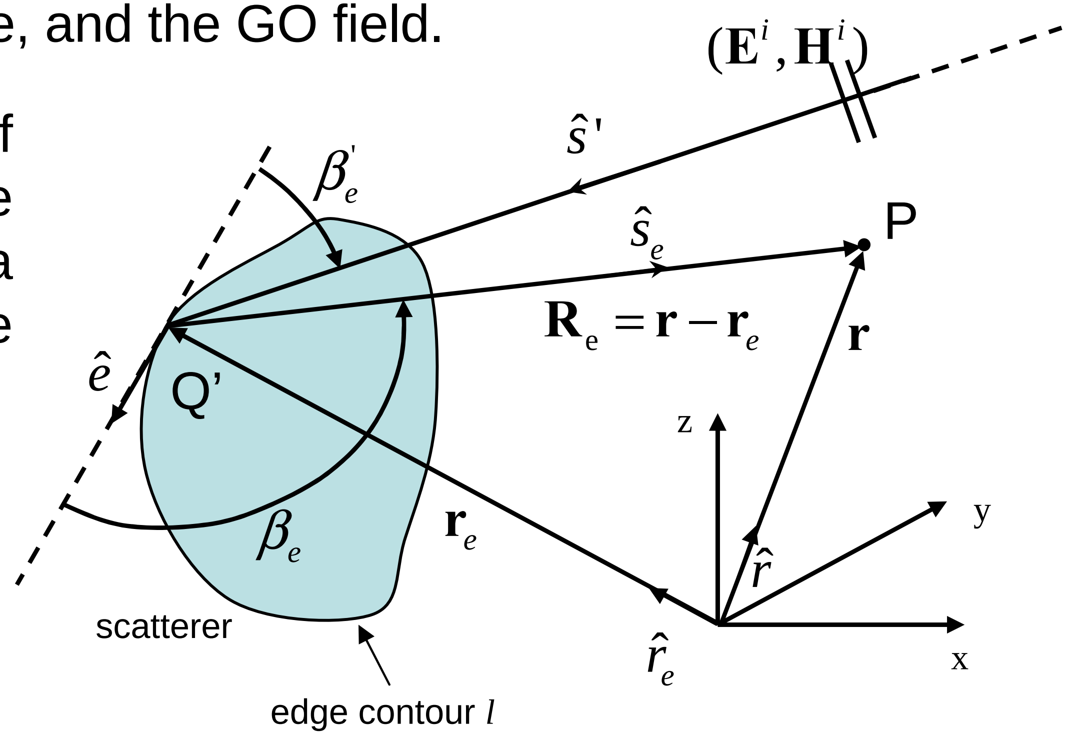
# INCREMENTAL THEORIES FOR EDGED BODIES

- The original idea is that an illuminating field undergoes a diffraction process at each incremental element of the edge (Young, Maggi, Sommerfeld).

- The observed "field pattern" (ex: diffracted field) is thus caused by the interaction between the "superposition" of elementary contributions distributed along the actual edge, and the GO field.

- The mathematical formulation of this "idea" may be asymptotically expressed as a line integral along the edge contour.

$$\mathbf{E}^{diff}(\mathbf{r}) \sim \int_l \mathbf{F}^{diff}(\mathbf{r}_e)\,dl$$

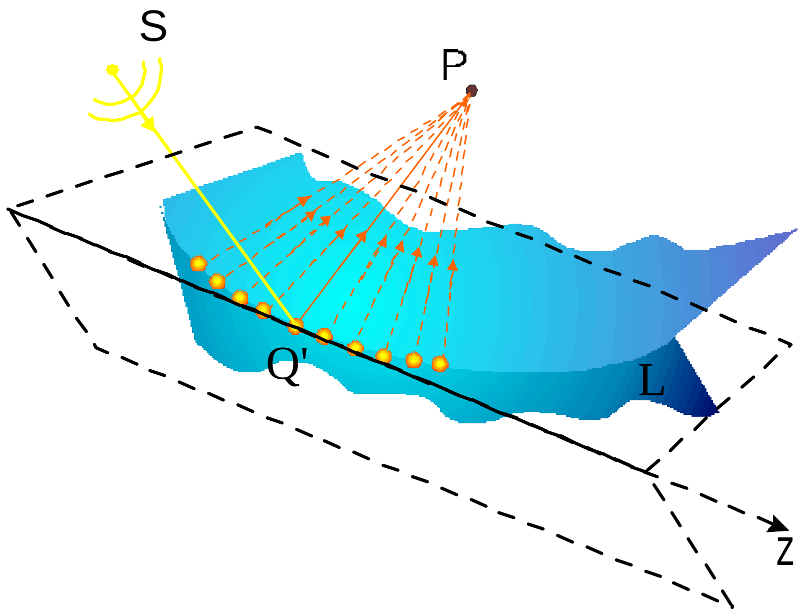


# INCREMENTAL THEORIES FOR EDGED BODIES

- At high frequency, each incremental field contribution $\mathbf{F}^{diff}(\mathbf{r}_e)$ is estimated from a local uniform, infinite, canonical configuration that more appropriately models the actual discontinuity.

$$\mathbf{E}^{diff}(\mathbf{r}) \sim \int_l \mathbf{F}^{diff}(\mathbf{r}_e)\,dl$$

$$\mathbf{F}_c^{diff}(\mathbf{r}_e)$$




- The two prevailing techniques to derive incremental contributions are the Current based and the Field based methods.

# CURRENT BASED METHODS

- Incremental fields contributions are deduce from the *currents* of local canonical problems.

- Incremental Length Diffraction Coefficients (ILDC):

  - K. M. Mitzner, Northrop Corp., Tech. Rep., Apr. 1974
  - R. A Shore and A. D Yaghjian, IEEE AP, Jan. 1988.
  - A. D. Yaghjian......

- Equivalent Edge Currents (EEC):

  - A. Michaeli, IEEE AP, Mar. 1984
  - A. Michaeli, IEEE AP, July,1986.

- Equivalent Edge Wave (EEW):

  - D. I. Bouturin and P Ya. Ufimtsev,  Sov. Phys. Acoust. Jul.-Aug. 1986.
  - P. Ya. Ufimtsev, Electromagnetics, Apr.-June 1991.

# FIELD BASED METHODS

- Incremental fields contributions are deduce from the *field* of local canonical problems.

- Incremental Diffracted Field:

  - A. Rubinowicz, Acta Phisica Polonica, 1965.

- Equivalent Current Method (ECM):

  - R. F. Millar, Proc. IEE , Mar. 1956.
  - C. E. Ryan and L. Peters, Jr., IEEE AP, May 1969.
  - E.F. Knott and T. B. A. Senior, IEEE AP, Sept.1973.

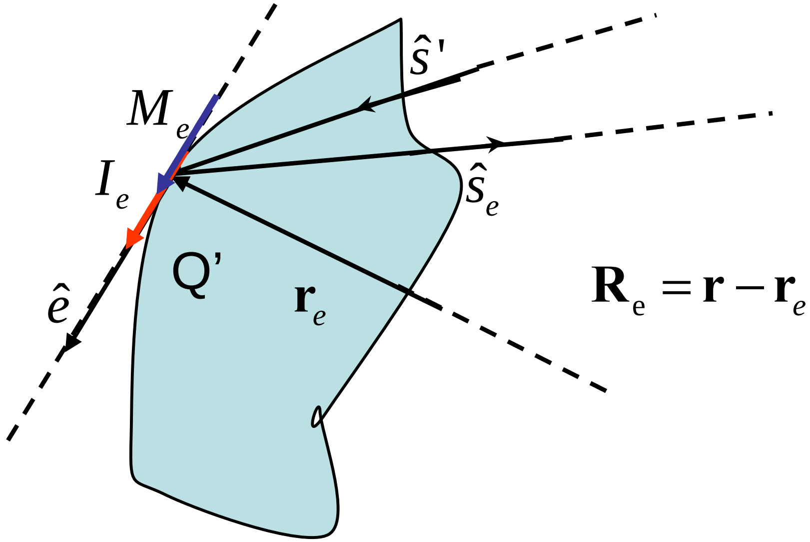
- Incremental Theory of Diffraction (ITD):

  - R. Tiberio, S. Maci, IEEE AP, May 1994.
  - R. Tiberio, S. Maci, and A. Toccafondi IEEE AP, 1996-2004.

# EQUIVALENT CURRENT METHOD

- The incremental field $\mathbf{F}_c^{diff}(\mathbf{r}_e)$ attributed to the edge element at $\mathbf{r}_e$ may be thought as generated by traveling-wave electric $I_e$ and magnetic $M_e$ line current, located at $\mathbf{r}_e$ and directed along the edge tangent $\hat{e}$ .

$$\mathbf{F}_c^{diff}(\mathbf{r}_e) = \frac{jk}{4\pi}[\hat{s}_e \times (\hat{s}_e \times \hat{e})\zeta I_e(Q') + (\hat{s}_e \times \hat{e})M_e(Q')]\frac{e^{-jk|\mathbf{R}_e|}}{|\mathbf{R}_e|}$$



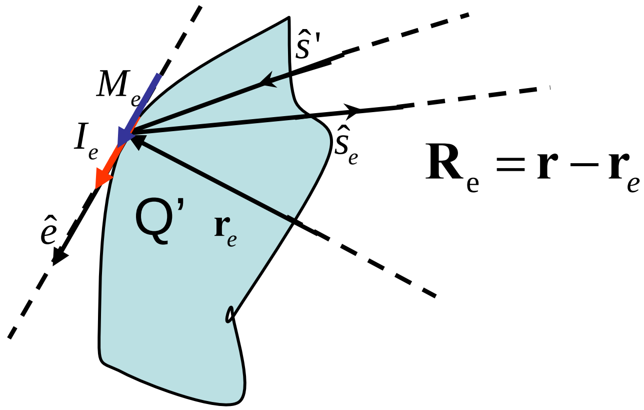

- The constant propagation factor is $k\cos\beta_e'$ .
- It assumed that the value of $I_e$ and $M_e$ depend linearly on the incident electric and magnetic field.

# EQUIVALENT CURRENT METHOD

- Each traveling-wave current radiates a conical wave

$$\hat{e}\cdot\mathbf{E}^{I}(P)\sim-\frac{k\zeta e^{j\pi/4}}{\sqrt{8\pi k}}I_e(Q')\sin\beta_e'\frac{e^{-jk|\mathbf{R}_e|}}{\sqrt{|\mathbf{R}_e|}}$$

$$\hat{e}\cdot\mathbf{H}^{M}(P)\sim-\frac{ke^{j\pi/4}}{\zeta\sqrt{8\pi k}}M_e(Q')\sin\beta_e'\frac{e^{-jk|\mathbf{R}_e|}}{\sqrt{|\mathbf{R}_e|}}$$




- If $\mathbf{R}_e \gg \lambda$ and $\mathbf{R}_e \ll \rho_e$ , the components of the GTD edge-diffracted field at P by a local canonical configuration at Q', can be approximated as:

$$\hat{e}\cdot\mathbf{E}^{d}(P)\sim[\hat{e}\cdot\mathbf{E}^{i}(Q')]D_s(\phi_e,\phi_e',\beta_e')\frac{e^{-jk|\mathbf{R}_e|}}{\sqrt{|\mathbf{R}_e|}}$$

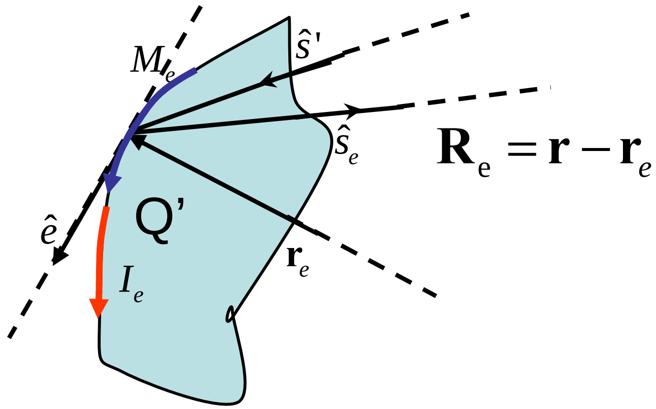
$$\hat{e}\cdot\mathbf{H}^{d}(P)\sim[\hat{e}\cdot\mathbf{H}^{i}(Q')]D_h(\phi_e,\phi_e',\beta_e')\frac{e^{-jk|\mathbf{R}_e|}}{\sqrt{|\mathbf{R}_e|}}$$

# EQUIVALENT CURRENT METHOD

- Comparing the above expressions outside the shadow boundary transition region:

$$I_e(Q')=-\frac{\sqrt{8\pi}e^{-j\pi/4}}{\zeta\sqrt{k}}\frac{[\hat{e}\cdot\mathbf{E}^i(Q')]}{\sin\beta_e'}D_s(\phi_e,\phi_e',\beta_e')$$

$$M_e(Q')=-\frac{\zeta\sqrt{8\pi}e^{-j\pi/4}}{\sqrt{k}}\frac{[\hat{e}\cdot\mathbf{H}^i(Q')]}{\sin\beta_e'}D_h(\phi_e,\phi_e',\beta_e')$$




- The current are “equivalent” since it depends on the angles of incidence and observation at the point of diffraction.
- Unlike GTD, the fields radiated by EC remain valid *even within the caustic regions* of edge-diffracted rays.
- These currents are defined only for observation points lying on the Keller cone of half-angle $\beta_e'$ .
- An heuristic extension to include points *outside* the Keller cone have been suggested (Knott and Senior).

$$I_e(Q')=-\frac{\sqrt{8\pi}e^{-j\pi/4}}{\zeta\sqrt{k}}\frac{[\hat{e}\cdot\mathbf{E}^i(Q')]}{\sqrt{\sin\beta_e'\sin\beta_e}}\tilde{D}_s(\phi_e,\phi_e',\beta_e',\beta_e)$$

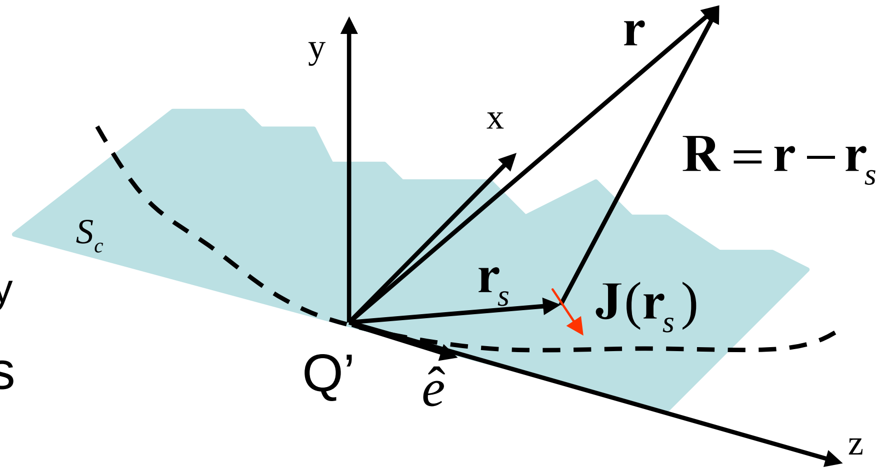
$$M_e(Q')=-\frac{\zeta\sqrt{8\pi}e^{-j\pi/4}}{\sqrt{k}}\frac{[\hat{e}\cdot\mathbf{H}^i(Q')]}{\sqrt{\sin\beta_e'\sin\beta_e}}\tilde{D}_h(\phi_e,\phi_e',\beta_e',\beta_e)$$

# EQUIVALENT EDGE CURRENT

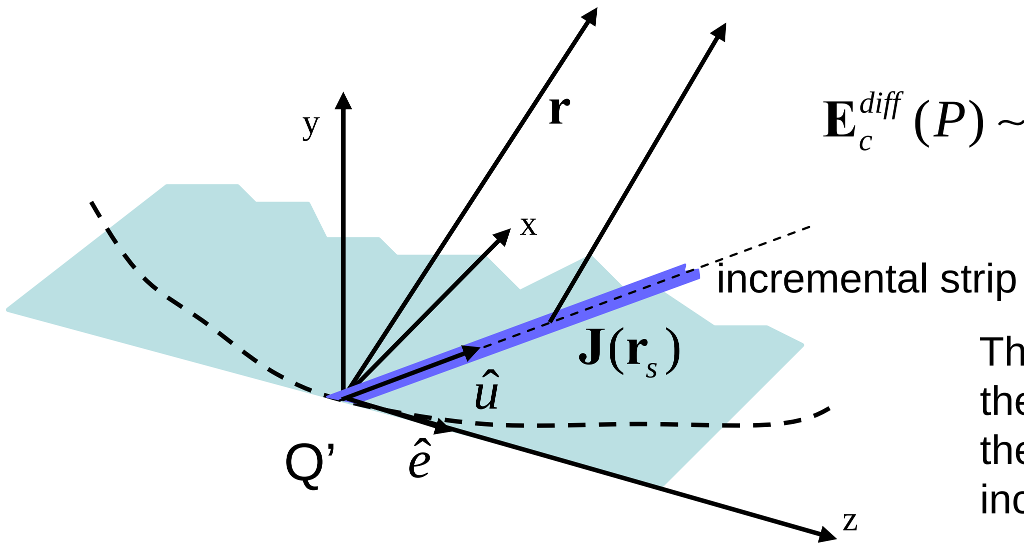
- The diffracted field by a local canonical configuration (pec) locally tangent at Q' is (half-plane):

$$\mathbf{E}_c^{diff}(P) \sim jk\zeta \iint_{S_c} \hat{\mathbf{R}} \times [\hat{\mathbf{R}} \times \boxed{\mathbf{J}(\mathbf{r}_s)}] \frac{e^{-jk|\mathbf{R}|}}{4\pi|\mathbf{R}|} dS$$

total induced electric current density




- The surface radiation integral is approximated in far field as:




$$\mathbf{E}_c^{diff}(P) \sim \int_{-\infty}^{\infty} jk\zeta(\hat{u} \times \hat{e}) \frac{e^{-jkr}}{4\pi r} \hat{r} \times \left[ \hat{r} \times \int_0^{\infty} \mathbf{J}(\mathbf{r}_s) e^{jk(\hat{r}\cdot\hat{u})u} du \right] dz$$

The incremental diffracted field associated to Q' is the end-point contribution to the field generated by the surface current density on an infinite extended incremental strip, starting at that edge point.

# EQUIVALENT EDGE CURRENT

- It is found that the equivalent edge current are

$$I(Q') = \frac{\hat{u}\times\hat{e}}{\left|\hat{r}\times\hat{e}\right|^2}\hat{r}\cdot(\hat{e}\times\hat{r})\times\int_0^\infty \mathbf{J}(\mathbf{r}_s)e^{jk(\hat{r}\cdot\hat{u})u}du$$

$$M(Q') = \zeta\frac{\hat{u}\times\hat{e}}{\left|\hat{r}\times\hat{e}\right|^2}\hat{e}\cdot\hat{r}\times\int_0^\infty \mathbf{J}(\mathbf{r}_s)e^{jk(\hat{r}\cdot\hat{u})u}du$$

- When the total current is applied, the above expressions provide an asymptotic approximation to the diffracted field.

$$I^{gtd}(Q') = L^I\{\mathbf{J}(\mathbf{r}_s)\}$$
$$M^{gtd}(Q') = L^M\{\mathbf{J}(\mathbf{r}_s)\}$$

- When the PO surface current density is applied, the above expressions provide an asymptotic approximation to the PO edge-diffracted field.

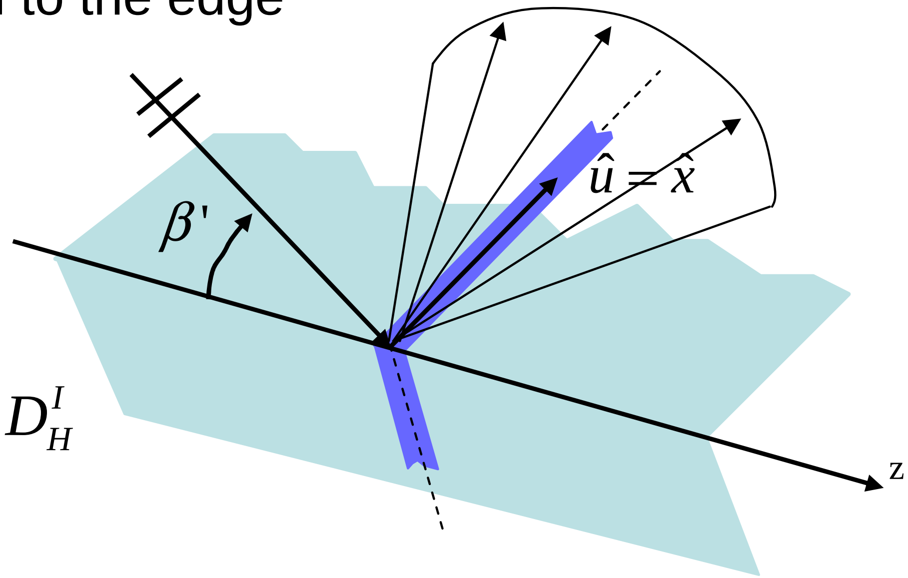
$$I^{po}_{ed}(Q') = L^I\{\mathbf{J}^{po}(\mathbf{r}_s)\}$$
$$M^{po}_{ed}(Q') = L^M\{\mathbf{J}^{po}(\mathbf{r}_s)\}$$

- When the FW (non uniform) surface current density is applied, the above expressions provide an asymptotic approximation to the PTD field.

$$I^{ptd}(Q') = L^I\{\mathbf{J}^{nu}(\mathbf{r}_s)\}$$
$$M^{ptd}(Q') = L^M\{\mathbf{J}^{nu}(\mathbf{r}_s)\}$$

# EQUIVALENT EDGE CURRENT

- Originally the strip was chosen normal to the edge




- Michaeli (1984) derived a set of EEC for the wedge diffracted field.

$$I = \left[ \mathbf{E}^i(Q') \cdot \hat{e} \right] \frac{1}{jk\zeta} D_E^I + \left[ \mathbf{H}^i(Q') \cdot \hat{e} \right] \frac{1}{jk} D_H^I$$

$$M = \left[ \mathbf{H}^i(Q') \cdot \hat{e} \right] \frac{\zeta}{jk} D_H^M$$

- Valid for arbitrary direction of observation.

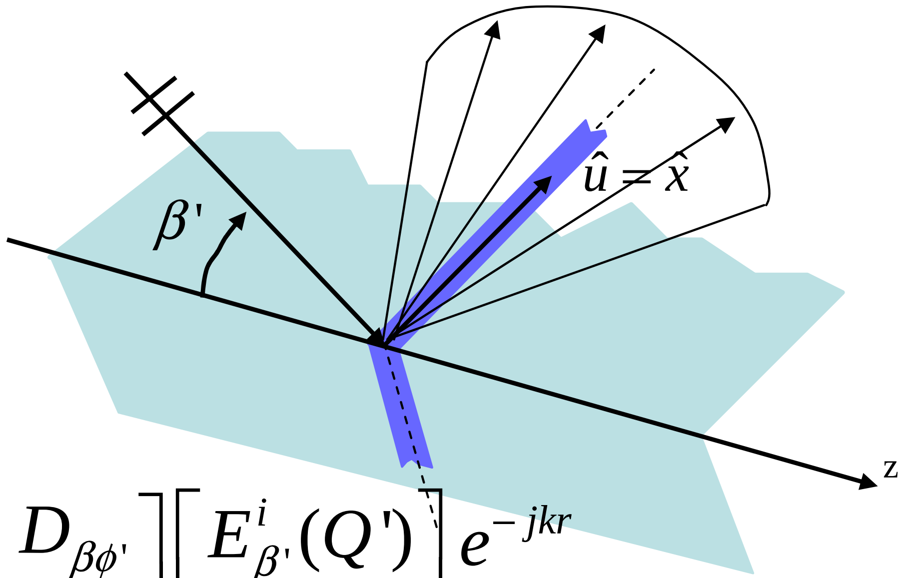
- The crosspolar component vanish at stationary point $\beta = \beta'$ .

- The incremental contributions are singular on a half-cone with tip at the edge, axis along the incremental strip and interior tip angle $\theta_p = \beta'$ .

# INCREMENTAL LEGTH DIFFRACTION COEFFICIENTS

- Mitzner (1984) first considered the *fringe wave surface current density* along the strips normal to the edge.

- He provided expressions in terms of Incremental Lenght Diffraction Coefficients (ILDC).




$$\mathbf{F}^{ptd}(Q') = \begin{bmatrix} F_{\beta}^{ptd} \\ F_{\phi}^{ptd} \end{bmatrix} = \begin{bmatrix} D_{\beta\beta'} & D_{\beta\phi'} \\ 0 & D_{\phi\phi'} \end{bmatrix} \begin{bmatrix} E_{\beta'}^{i}(Q') \\ E_{\phi'}^{i}(Q') \end{bmatrix} \frac{e^{-jkr}}{4\pi r}$$

- The ILDC's are singular on a half-cone with tip at the edge, axis along the incremental strip and interior tip angle $\theta_p = \beta'$.

- It is found (Knott, 1985) that if this coefficients is cast in the form of EEC, the two methods are equivalents (they differ by terms due to PO).

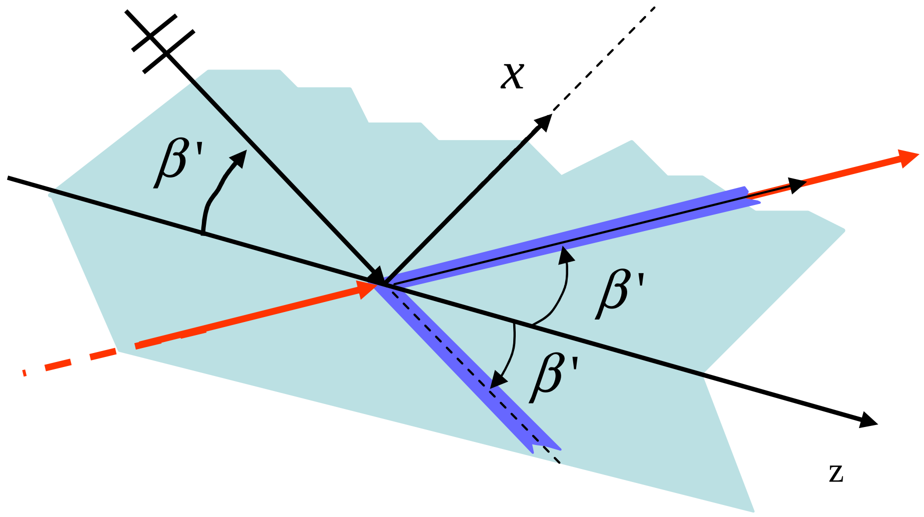
# REDUCTION OF SINGULARITIES

- In general, for arbitrary strip orientation, the singularities lie on a half-cone along the strip with interior angle:

$$\theta_p = 2\arccos\left[(\cos\beta'\hat{x} + \sin\beta'\hat{z})\cdot\hat{u}\right]$$

- To reduce the number of singular directions, Michaeli (1986) and Ufimtsev (1991) considered the elementary strip in the direction of the "grazing diffracted ray".

$$(\cos\beta'\hat{x} + \sin\beta'\hat{z}) = \hat{u} \rightarrow \theta_p = 0$$




- The resulting incremental fields are singular only along the strip direction, when the incident field is grazing.

# PTD-EEC

- Michaeli derived a set EEC by asymptotic evaluation of the fringe wave current radiation integral for the canonical wedge solution.

$$I^f = E_t^i \frac{2jY}{k \sin^2 \beta'} \frac{\sqrt{2} \sin(\phi'/2)}{\cos \phi' + \mu} [\sqrt{1-\mu} - \sqrt{2} \cos(\phi'/2)]$$

$$+ H_t^i \frac{2j}{k \sin \beta'} \frac{1}{\cos \phi' + \mu}$$

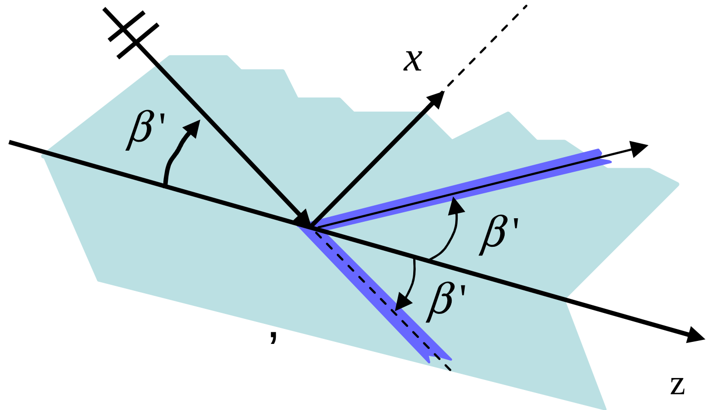
$$\cdot [\cot \beta' \cos \phi' + \cot \beta \cos \phi$$

$$+ \sqrt{2} \cos(\phi'/2)(\mu \cot \beta' - \cot \beta \cos \phi)(1-\mu)^{-1/2}],$$

$$M^f = H_t^i \frac{2jZ \sin \phi}{k \sin \beta \sin \beta'} \frac{1}{\cos \phi' + \mu}$$

$$\cdot \left[1 - \frac{\sqrt{2} \cos(\phi'/2)}{\sqrt{1-\mu}}\right]$$

(Half-Plane)




$$\mu = \frac{\sin \beta \sin \beta' \cos \phi + \cos \beta (\cos \beta' - \cos \beta)}{\sin^2 \beta'}$$

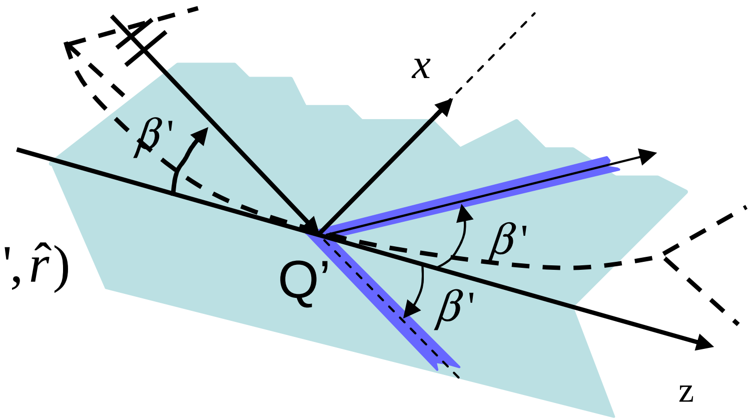
- It exhibits a **non-reciprocal** behavior w.r.t. the aspects of incidence and observation.

# PTD-EEW

- Ufimtsev (1991) derived a set PTD incremental diffraction coefficients based on the concept of Elementary Edge Wave.
- EEWs are the waves scattered "…by the vicinity of an infinitesimal element of a curved edge."
- They are calculated as the spherical edge waves radiated by the nonuniform currents flowing along the elementary strips
- High-frequency asymptotics are provided in the form:

$$\mathbf{F}_E^{ptd}(P) = \frac{1}{2\pi} D_e(Q',\hat{r}) \frac{e^{-jkr}}{r} \quad ; \quad \mathbf{F}_H^{ptd} = \frac{1}{\zeta}\hat{r}\times\mathbf{F}_E^{ptd}$$

$$D_e(Q',\hat{r}) = \left[\mathbf{E}^i(Q')\cdot\hat{e}\right]\mathbf{M}(Q',\hat{r}) + \zeta\left[\mathbf{H}^i(Q')\cdot\hat{e}\right]\mathbf{N}(Q',\hat{r})$$




- It exhibits a **non-reciprocal** behavior w.r.t. the aspects of incidence and observation.

# OTHER CONTRIBUTIONS

- Shore and Yaghjian (1989) developed a simple substitution method for obtaining ILDC's directly from the 2-D scattered far fields of PEC scatterer.

- It allows to obtain the ILDCs without any integration, or differentiation or specific knowledge of the currents.

- Ando (1991) introduced a set of GTD equivalent edge currents in order to completely eliminates the false singularities, in which the orientation of the strip depends on the observation aspects.

- ILDC's for aperture (Michaeli 1995), strips and slits (Yaghjian 1997) as well as for cylindrical scatterer (Yaghjian 2001) have also been proposed.

# INCREMENTAL THEORY OF DIFFRACTION

- Explicit ITD formulations have been obtained for:
  - Pec local wedges.
  - Pec local thin circular cylinders.
  - Double local edges

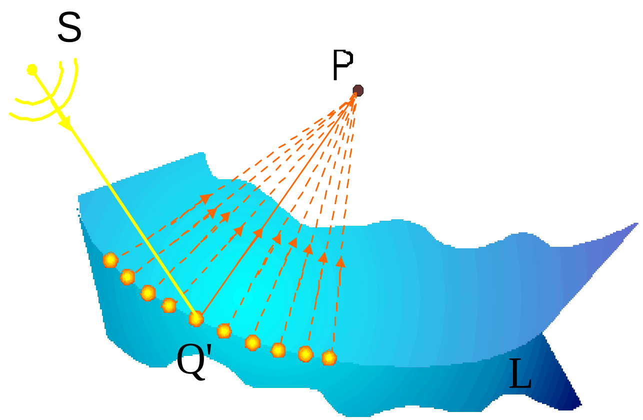
- Heuristic ITD formulations have been obtained for:
  - Local edges in impedance planar surfaces.
  - Local edges in thin coated and dielectric panels.

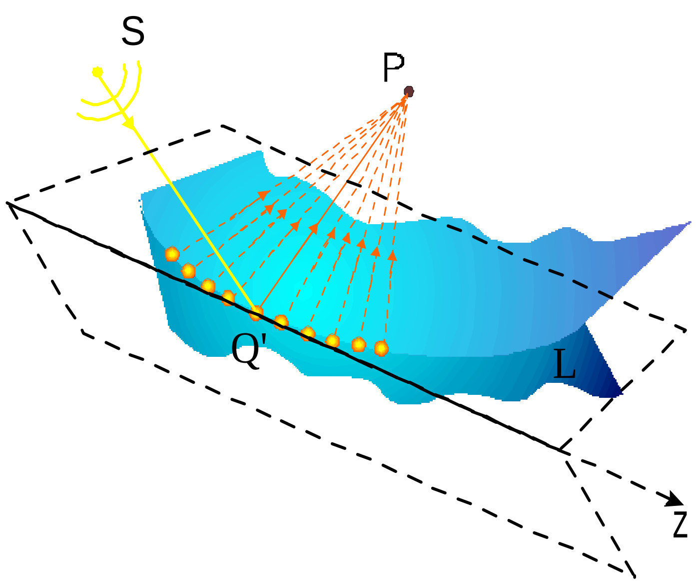
# ITD LOCALIZATION PROCESS

- Diffracted field

$$\mathbf{E}^{diff}(P) = \int_L \mathbf{F}^{diff}(Q')dl$$

the incremental field can be deduced from an appropriate local canonical problem




- Find a convenient representation for the diffracted field of the canonical problem.

$$\mathbf{E}_c^{diff}(P) = \int_{-\infty}^{+\infty} \mathbf{F}_c^d(z'')dz''$$

- Assume that at high-frequency

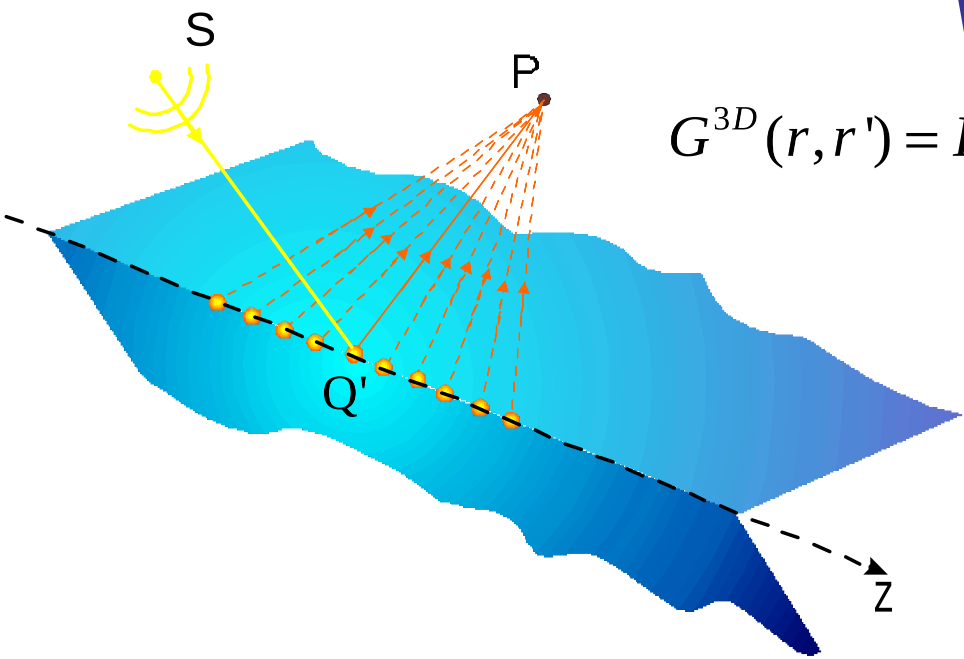
$$\boxed{\mathbf{F}^{diff}(Q') = \mathbf{F}_c^d(z'')\Big|_{z''=0}}$$

# A USEFUL REPRESENTATION

- For an object, with a uniform cylindrical canonical configuration along the *z*-axis, by spectral synthesis:

$$G^{3D}(r,r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^{2D}(\rho,\rho',k_\rho^{"})e^{-jk_z^{"}(z-z')}dk_z^{"}$$

- Reciprocity suggests




$$G^{3D}(r,r') = L\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} U(k_\rho^{"},\rho)\cdot U(k_\rho^{"},\rho')e^{-jk_z^{"}(z-z')}dk_z^{"}\right]$$

- It is found that $G^{2D}$ may be represented as the result of the application of a linear operator L[].

# INCREMENTAL FIELD CONTRIBUTION

$$G^{3D}(r,r') = L\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} U(k_\rho^{"},\rho)\cdot U(k_\rho^{"},\rho')e^{-jk_z^{"}(z-z')}dk_z^{"}\right]$$

- By Fourier analysis, this spectral integral representation is interpreted as the *spatial* convolution product of two functions:

$$G^{3D}(r,r') = L\left[\int_{-\infty}^{\infty} u(z"-z,\rho)\cdot u(z'-z",\rho')dz"\right]$$

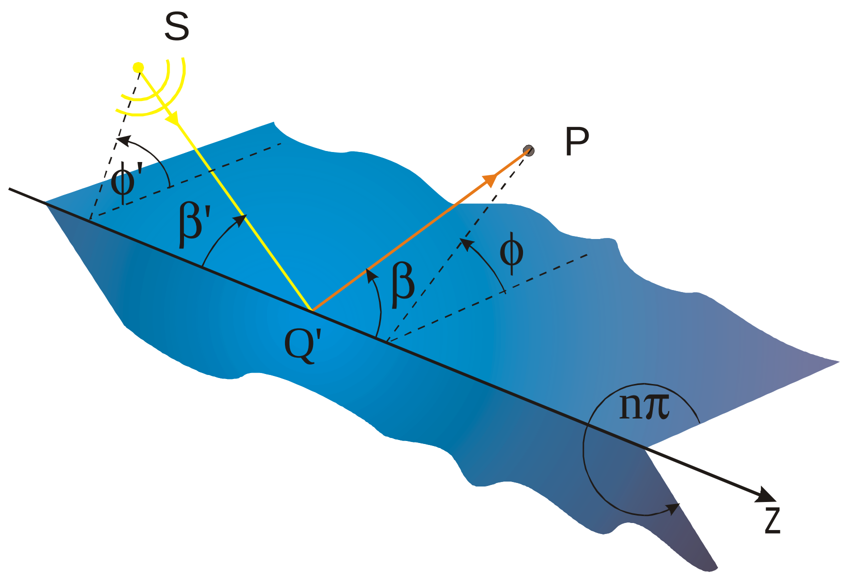
- This above spatial integral representation allows to directly define the local incremental field contribution

$$\mathrm{F}_{\mathrm{c}}^{\mathrm{d}}(z")\Big|_{z"=0} = u(-z,\rho)\cdot u(z',\rho') = F^{-1}\left[U(k_\rho,\rho)\right]\cdot F^{-1}\left[U(k_\rho^{'},\rho')\right]$$

Inverse Fourier transform

# WEDGE-SHAPED CONFIGURATIONS

- This "FT-Convolution" process is directly applicable to the exact solution for the wedge




$$G^{3D}(r,r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^{2D}(\rho,\rho',k_\rho^{"})e^{-jk_z^{"}(z-z')}dk_z^{"}$$

- Plane wave spectral representation

$$G^{2D}(\rho,\rho';k_\rho^{"}) = \frac{1}{2\pi}\int_{C_\alpha}\int_{C_{\alpha'}} G^e(\alpha,\alpha';\phi,\phi')e^{-jk_\rho^{"}\rho\cos\alpha}e^{-jk_\rho^{"}\rho'\cos\alpha'}d\alpha d\alpha'$$

- ITD diffracted field contribution

$$dF_c^d(Q') = \frac{k^2}{2(j2\pi)^2}\int_{C_\alpha}\int_{C_{\alpha'}} G^e(\alpha,\alpha';\phi,\phi')u(-z,\rho)u(z',\rho')d\alpha d\alpha'$$

$$u(z',\rho') = \frac{1}{2\pi}\int_{-\infty}^{\infty} k_\rho^{'}e^{-jk_\rho^{'}\rho\cos\alpha}e^{jk_z^{'}z'}dk_z^{'}$$

# ASYMPTOTIC ANALYSIS (I)

- Spectral representation (EM case)

$$dF_c^d(Q_l) = \int_{C_\alpha}\int_{C_{\alpha'}}\int_{C_\theta}\int_{C_{\theta'}} G^e[(\alpha-\alpha'),\phi,\phi']e^{-jk[rf(\alpha,\theta,\beta)+r'f(\alpha',\theta',\beta')]}\sin\theta\sin\theta' d\theta d\theta' d\alpha d\alpha'$$

$$f(\alpha^{(')},\theta^{(')},\beta^{(')}) = \sin\beta^{(')}\sin\theta^{(')}\cos\alpha^{(')} + \cos\beta^{(')}\cos\theta^{(')} = \cos\gamma^{(')}$$

- The desired stationary phase condition $\cos\gamma^{(')} = 1$ is achieved at

$$\cos\alpha = \frac{1-\cos\theta\cos\beta}{\sin\theta\sin\beta} \qquad \cos\alpha' = \frac{1-\cos\theta'\cos\beta'}{\sin\theta'\sin\beta'}$$

- Two more condition are required

# ASYMPTOTIC ANALYSIS (II)

- In the asymptotic analysis, the structure of the spectrum (spectral synthesis) requires.

$$\upsilon = (\alpha - \alpha') \quad \text{independent of} \quad \theta, \theta'$$

- This is achieve d by $\theta = \theta'$( analytic continuation of the diffraction cone).
- The value of $\upsilon = (\alpha - \alpha')$ at the stationary point is:

$$\cos(\alpha - \alpha') = \frac{1 - \cos\beta\cos\beta'}{\sin\beta\sin\beta'}$$

$$\sin(\alpha - \alpha') = j\frac{\cos\beta - \cos\beta'}{\sin\beta\sin\beta'}$$

**(Rubinowicz)**

- Also

$$\sin^2\theta = \sin\beta\sin\beta'$$

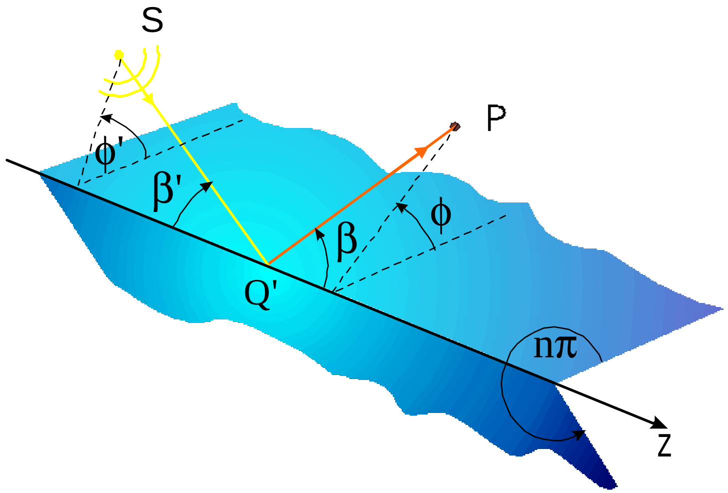
# ITD SOLUTION FOR WEDGE PROBLEM

$$\mathbf{F}_c^d(Q') = \begin{bmatrix} F_\beta^d \\ F_\phi^d \end{bmatrix} = \begin{bmatrix} D_e(\upsilon,\phi,\phi') & 0 \\ 0 & D_h(\upsilon,\phi,\phi') \end{bmatrix} \begin{bmatrix} E_{\beta'}(Q') \\ E_{\phi'}(Q') \end{bmatrix} \frac{e^{-jkr}}{2\pi r}$$

$$D_{e,h}(\upsilon,\phi,\phi') = D_n[\upsilon,\Phi^-](-,+)D_n[\upsilon,\Phi^+]$$

$$D_n[\upsilon,\Phi^\pm] = d_n[\upsilon,\Phi^\pm] + d_n[\upsilon,-\Phi^\pm]$$

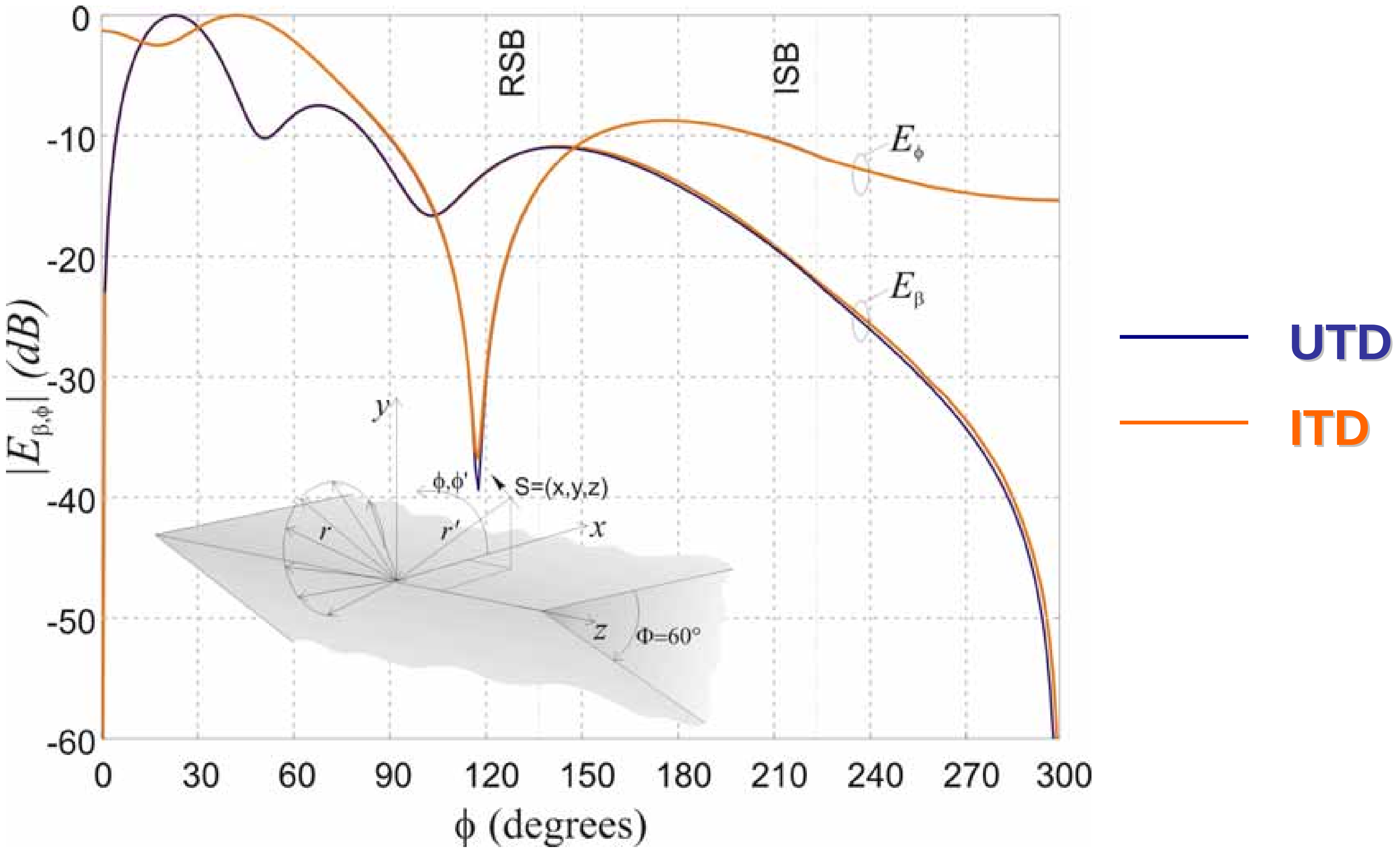
$$d_n[\upsilon,\chi] = \frac{1}{2n} \frac{\sin\left(\dfrac{\pi-\chi}{n}\right)}{\cos\left(\dfrac{\upsilon}{n}\right) - \cos\left(\dfrac{\pi-\chi}{n}\right)}$$




- ITD stationary phase point $\cos\upsilon = \dfrac{1-\cos\beta\cos\beta'}{\sin\beta\ \sin\beta'}$ $\quad \Phi^\pm = \phi - \phi'$
- explicitly satisfies reciprocity
- well-behaved at any aspect, including observation points lying on the axis of the local canonical wedge.

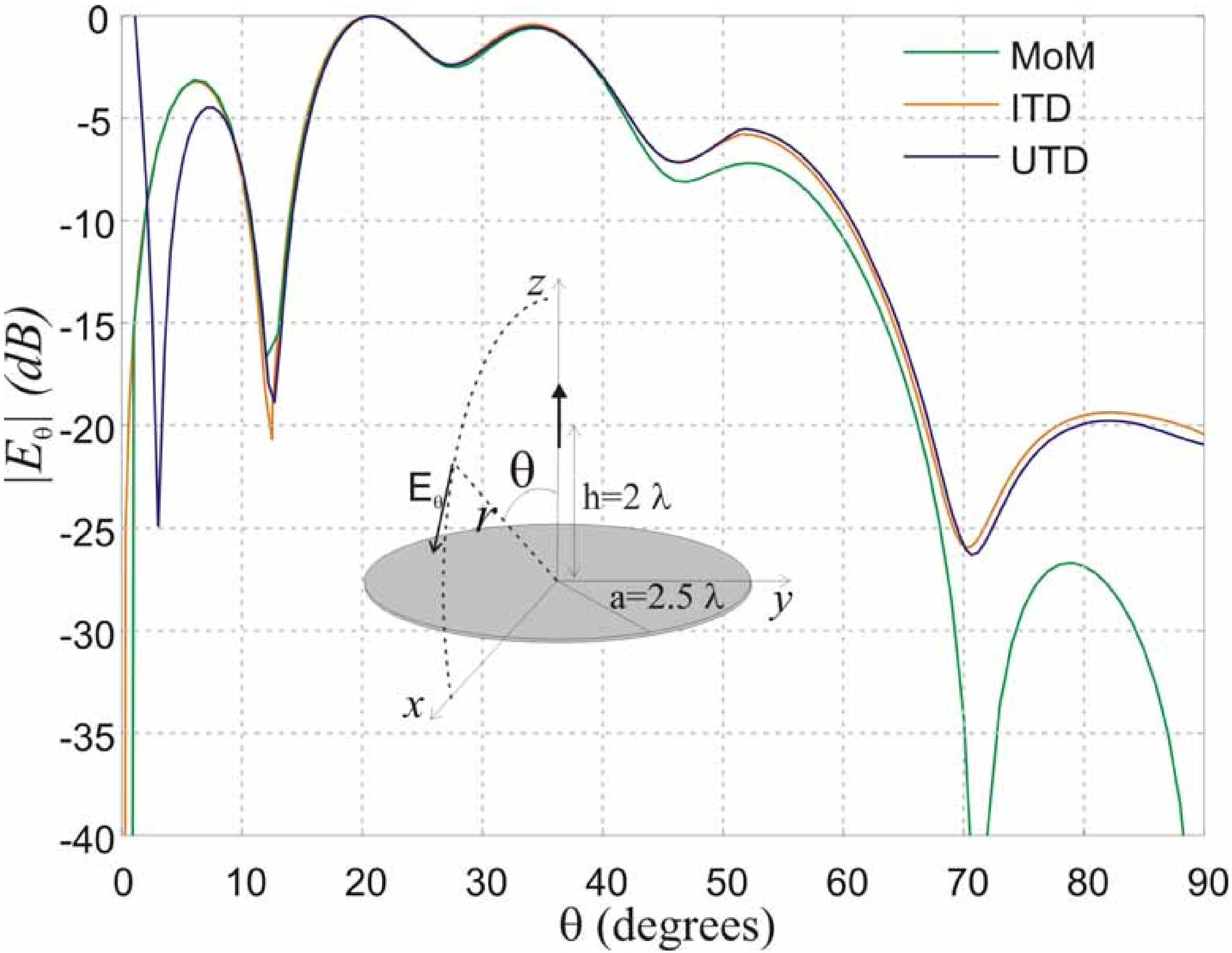
# ITD vs. UTD

- When a stationary phase condition has been well established, the integration of ITD incremental fields smoothly blends into the field predicted by the UTD.



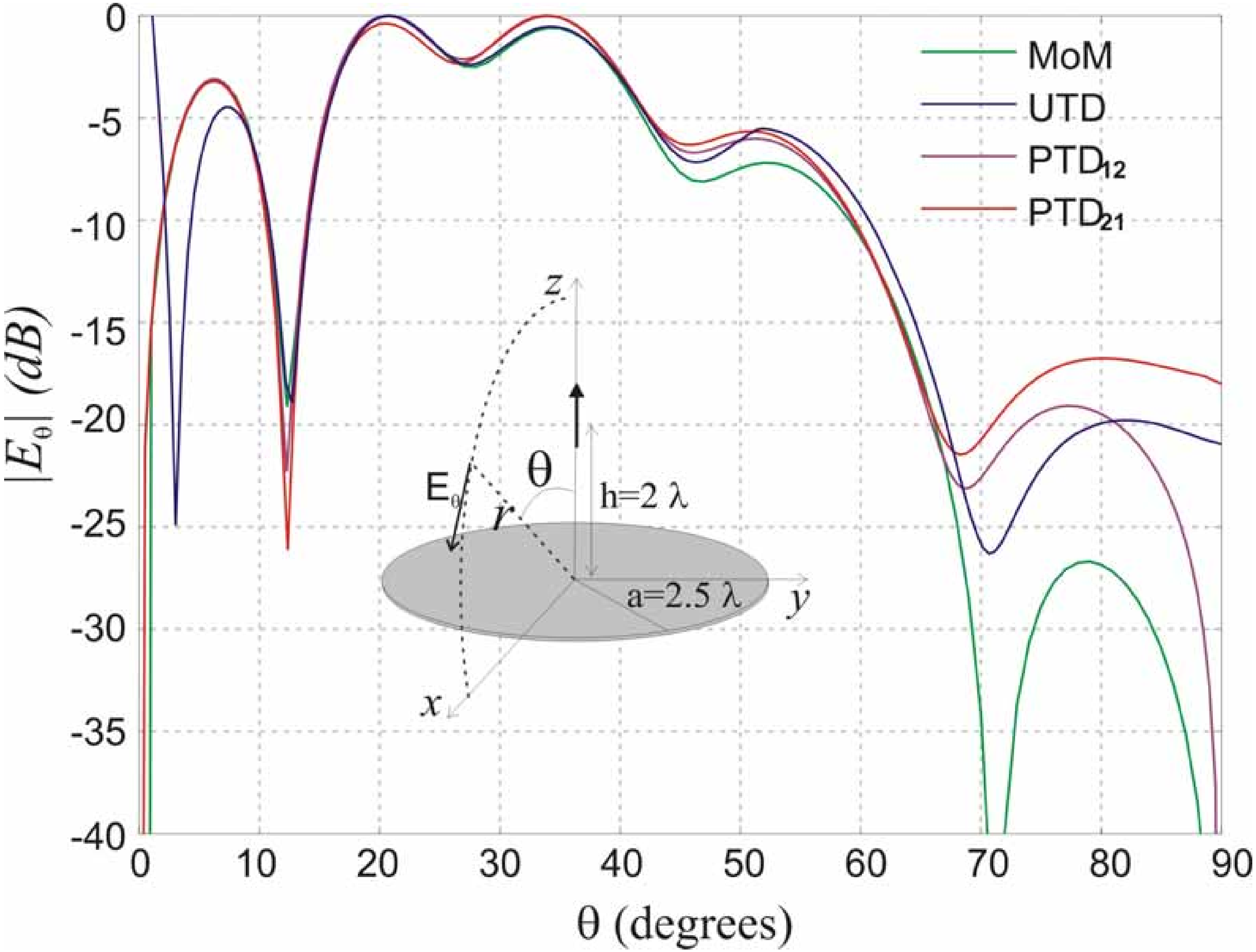
# ITD vs. MOM, UTD

- Far-zone pattern of a circular disc



# PTD vs. MOM, UTD

- Far-zone pattern of a circular disc



# TWO ALTERNATIVE APPROACHES

- The GO ray field is well-behaved within its domain of existence, the total field is

$$\mathbf{E} = \mathbf{E}^{ggo} + \int_l \mathbf{F}^d(Q')dl$$

$$\mathbf{E}^{ggo} = \mathbf{E}^i + \mathbf{E}^{po} - \mathbf{E}_d^{po} \sim \mathbf{E}^{go}$$

$$\mathbf{E}^{diff} \sim \mathbf{E}^{utd}$$

- The GO ray field regime has not been well established or when an augmentation of the PO field is desired

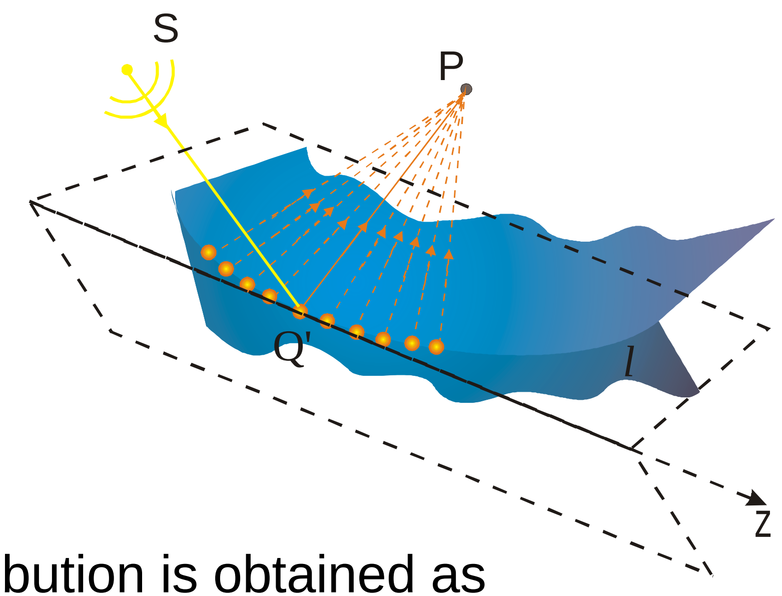
$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^{po} + \int_l \left[\mathbf{F}^d(Q') - \mathbf{F}_d^{po}(Q')\right] dl$$

$$\mathbf{E}^{fw} \sim \mathbf{E}^{ptd} = \mathbf{E}^{gtd} - \mathbf{E}_{ed}^{po}$$

- It is more convenient from numerical point of view

- Note: the fringe contributions are different from those of PTD, only the terminology is the same

# ITD FRINGE FORMULATION (I)




$$\mathbf{E}^{s} = \mathbf{E}^{po} + \int_{l} \mathbf{F}^{f}(Q')dl$$

- Well-behaved at any aspect
- Tends to remedy the typical non-reciprocity of PO.
- A localized incremental, fringe field contribution is obtained as

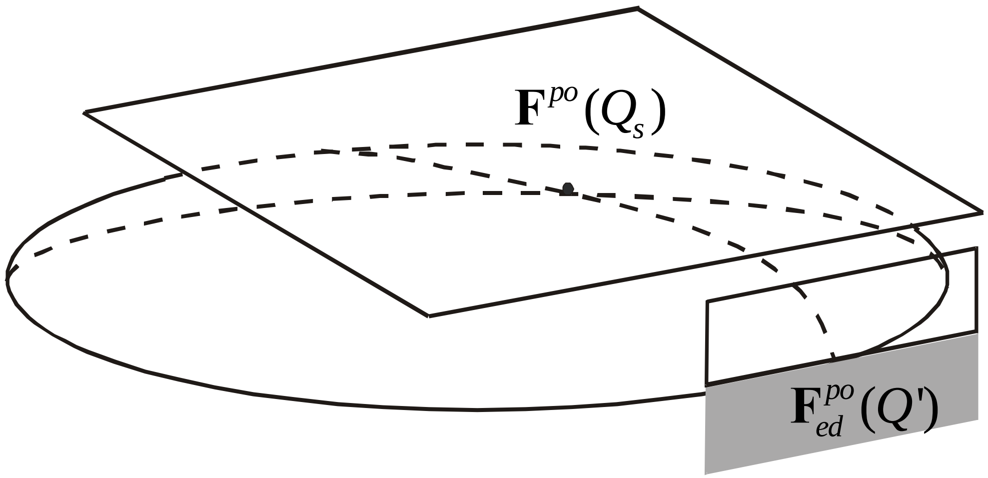
$$\mathbf{F}^{f}(Q') \sim \mathbf{F}_{c}^{f}(Q') = \mathbf{F}_{c}^{d}(Q') - \mathbf{F}_{ed}^{po}(Q')$$

- The incremental PO edge-diffracted field $\mathbf{F}_{ed}^{po}(Q')$ is obtained by applying the ITD localization process to a local canonical problem with PO currents.

# PO EDGE-DIFFRACTED FIELD

- The PO edge-diffracted field is obtained by integration of the incremental PO end-point contributions

$$\mathbf{E}_{ed}^{po} = \int_{l} \mathbf{F}_{ed}^{po}(Q')dl$$




- They are found by applying the ITD localization process to the local canonical problem of a plane half-lit and half-shadowed

# INCREMENTAL PO EDGE-DIFFRACTED FIELD

$$\mathbf{F}_{ed}^{po}(Q') = \begin{bmatrix} F_{\beta}^{po} \\ F_{\phi}^{po} \end{bmatrix} = \begin{bmatrix} D_{11}^{po}(\upsilon,\phi,\phi') & D_{12}^{po}(\upsilon,\phi,\phi') \\ 0 & D_{22}^{po}(\upsilon,\phi,\phi') \end{bmatrix} \cdot \begin{bmatrix} E_{\beta'}(Q_l) \\ E_{\phi'}(Q_l) \end{bmatrix} \frac{e^{-jkr}}{2\pi r}$$

$$D_{11}^{po}(\upsilon;\phi,\phi') = d_1[\upsilon,\Phi^-] - d_1[\upsilon,\Phi^+]$$

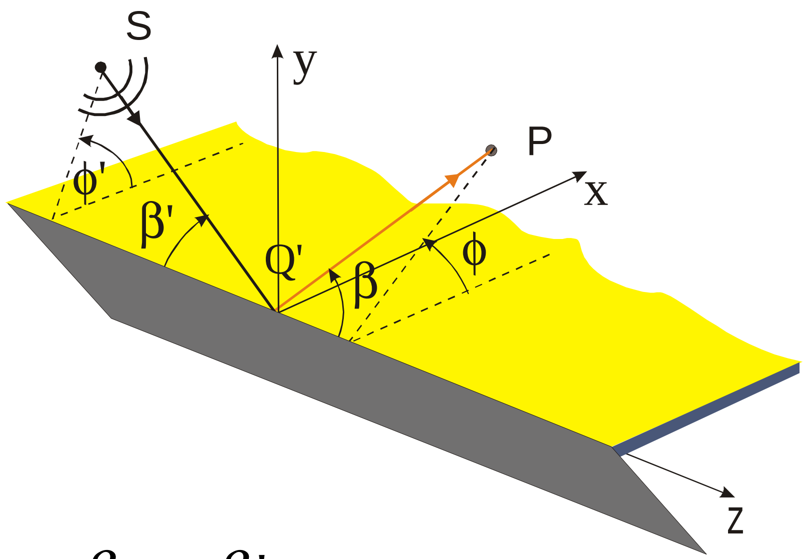
$$\Phi^{\pm} = \phi - \phi'$$

$$D_{22}^{po}(\upsilon;\phi,\phi') = d_1[\upsilon,\Phi^-] + d_1[\upsilon,\Phi^+]$$

$$D_{12}^{po}(\upsilon;\phi,\phi') = \cos\theta$$

$$d_1[\upsilon,\Omega] = \frac{1}{2}\frac{\sin\Omega}{\cos\upsilon + \cos\Omega}$$

$$\cos\theta = sgn(z+z')\sqrt{1-\sin\beta\sin\beta'}$$




- ITD stationary phase point $\cos\upsilon = \dfrac{1-\cos\beta\cos\beta'}{\sin\beta\ \sin\beta'}$

# ITD FRINGE FORMULATION FOR A LOCAL HALF-PLANE

$$\mathbf{F}_c^f(Q') = \begin{bmatrix} F_\beta^f \\ F_\phi^f \end{bmatrix} = \begin{bmatrix} D_{11}^f(\upsilon,\phi,\phi') & D_{12}^f(\upsilon,\phi,\phi') \\ 0 & D_{22}^f(\upsilon,\phi,\phi') \end{bmatrix} \cdot \begin{bmatrix} E_{\beta'}(Q') \\ E_{\phi'}(Q') \end{bmatrix} \frac{e^{-jkr}}{2\pi r}$$

$$D_{11}^f(\upsilon;\phi,\phi') = d_2^f[\upsilon,\Phi^-] - d_2^f[\upsilon,\Phi^+]$$

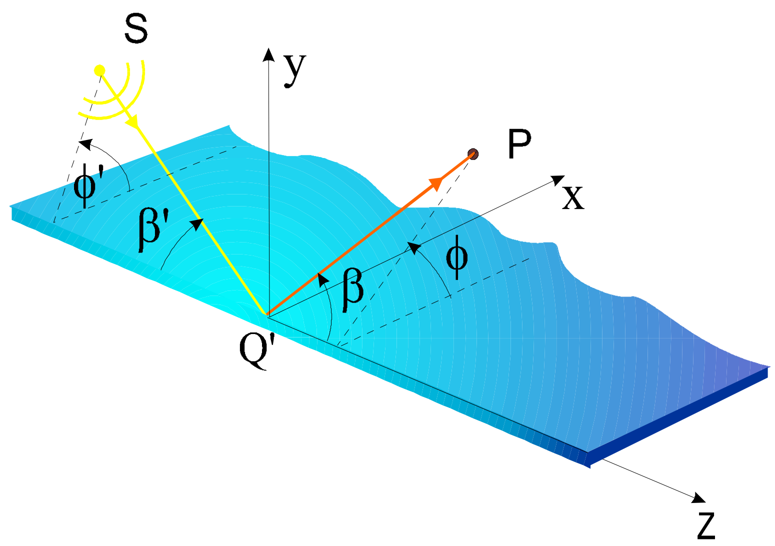
$$\Phi^\pm = \phi - \phi'$$

$$D_{22}^f(\upsilon;\phi,\phi') = d_2^f[\upsilon,\Phi^-] + d_2^f[\upsilon,\Phi^+]$$

$$D_{12}^f(\upsilon;\phi,\phi') = -\cos\theta$$

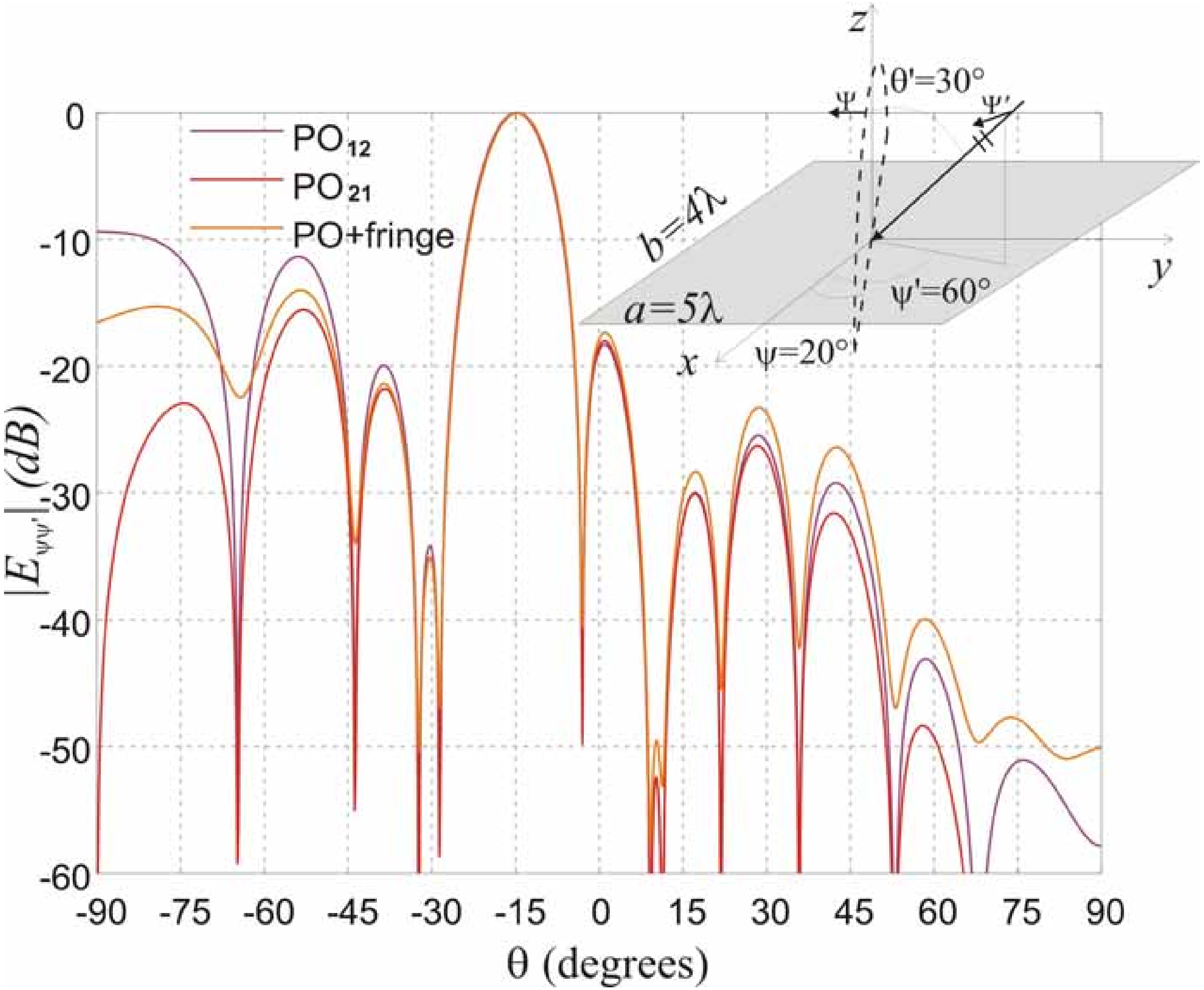
$$d_2^f[\upsilon,\Omega] = \frac{1}{2}\frac{\cos(\Omega/2)}{\cos(\upsilon/2)+\sin(\Omega/2)}$$

$$\cos\theta = sgn(z+z')\sqrt{1-\sin\beta\sin\beta'}$$



# FRINGE ITD vs. PTD

- ITD fringe augmentation tends to remedy the typical non-reciprocity of current based methods



# INCREMENTAL THEORY OF DIFFRACTION

# Circular Cylinders

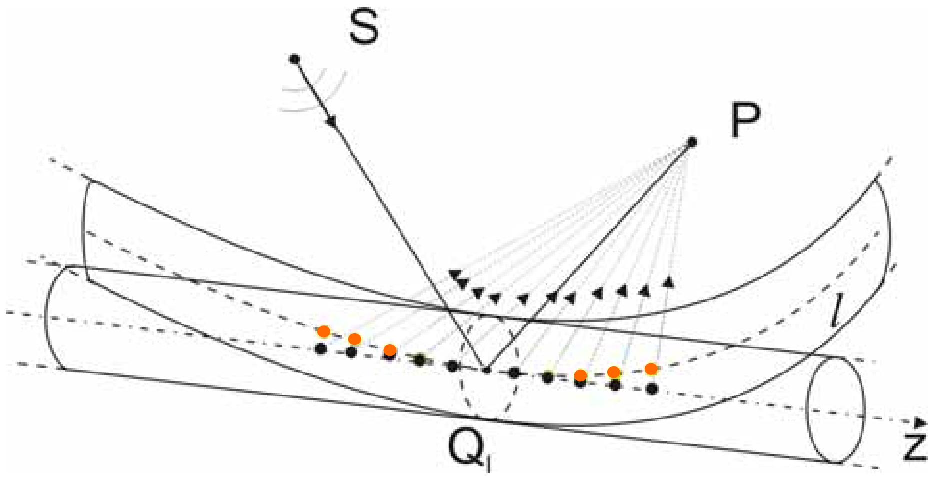
# INCREMENTAL SCATTERED FIELD CONTRIBUTION

- Total field (scalar formulation)

$$\Psi^t(P) = \Psi^i(P) + \Psi^s(P)$$

- Scattered field

$$\Psi^s(P) = \int_l \psi(Q_l)\,dl$$




- The incremental field may be deduced from an appropriate local canonical problem tangent at $Q_l$. To this purpose, we need to find the convenient field representation for the local canonical problem

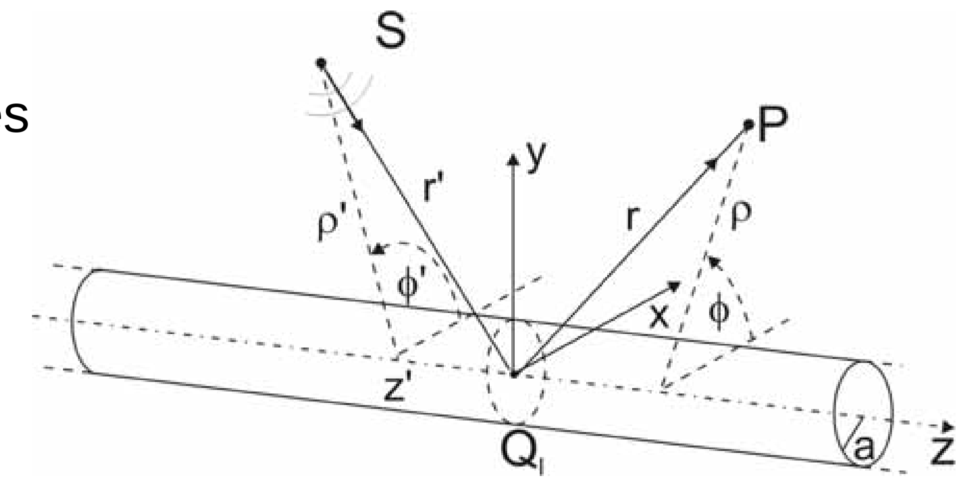
$$\Psi_c^s(P) = \int_{-\infty}^{\infty} \psi_c(z)\,dz$$

- Then, at high-frequency, it is assumed that

$$\psi(Q_l) = \psi_c(z)\Big|_{z=0}$$

# LOCAL CANONICAL PROBLEM

- Canonical solution (spectral synthes




$$\Psi_c^s(r,r') = -\frac{1}{8\pi j}\sum_{n=-\infty}^{\infty} e^{-jn(\phi-\phi')}\,\Psi_n^s(r,r')$$

$$\psi_n^{e,h}\left(\mathrm{r},\mathrm{r}'\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_n^{e,h}\left(k_\rho'' a\right) H_n^{(2)}\left(k_\rho''\rho\right) H_n^{(2)}\left(k_\rho''\rho'\right) e^{-jk_z''(z-z')}\,dk_z''$$

$$G_n^{e,h}(\rho,\rho',k_\rho'')$$

where $c_n^{(e,h)}(k_\rho'' a) = \begin{cases} \dfrac{J_n(k_\rho'' a)}{H_n^{(2)}(k_\rho'' a)} & \text{soft b.c. } (e) \\[2ex] \dfrac{J'_n(k_\rho'' a)}{H'^{(2)}_n(k_\rho'' a)} & \text{hard b.c. } (h) \end{cases}$ and $k_\rho'' = \sqrt{k^2 - (k_z'')^2}$

# THE APPROPRIATE SPECTRUM FUNCTIONS (I)

- Soft boundary conditions

$$2c_n^e(k_\rho''a) = 2\frac{J_n(k_\rho''a)}{H_n^{(2)}(k_\rho''a)} = 1 + \frac{H_n^{(1)}(k_\rho''a)}{H_n^{(2)}(k_\rho''a)} = 1 + \sigma_n^e(k_\rho''a)\,\sigma_n^e(k_\rho''^{*}a)$$

$$\sigma_n^e(k_\rho''a) = \frac{H_n^{(1)}(k_\rho''a)}{\left|H_n^{(2)}(k_\rho''a)\right|}$$

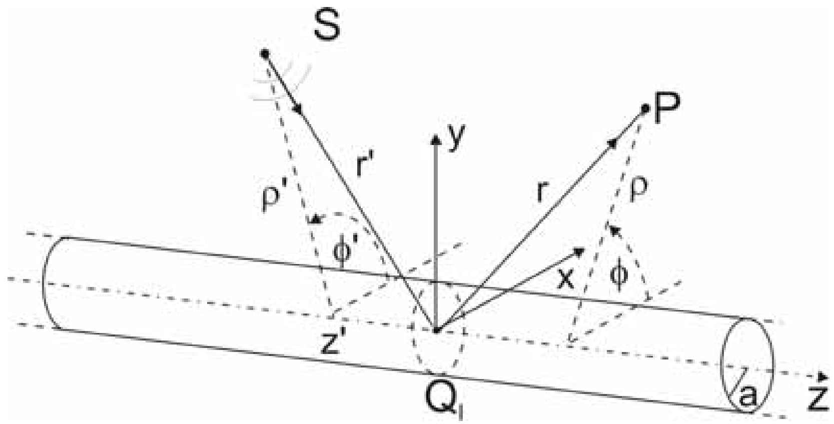
for $k_\rho$ real $\quad \dfrac{H_n^{(1)}(k_\rho a)}{H_n^{(2)}(k_\rho a)} = e^{j2\vartheta_n(k_\rho a)} \longrightarrow \sigma_n^e(k_\rho a) = e^{j\vartheta_n(k_\rho a)}$

- Thus, the typical n-th term in the integrand of the exact solution is

$$G_n^{e,h}(\rho,\rho',k_\rho'') = \frac{1}{2}\left[1 + \sigma_n^e(k_\rho''a)\,\sigma_n^e(k_\rho''^{*}a)\right] H_n^{(2)}(k_\rho''\rho)\,H_n^{(2)}(k_\rho''\rho')$$

# THE APPROPRIATE SPECTRUM FUNCTIONS (II)

- This leads to the useful representation




$$G_n(\rho,\rho',k_\rho)=\frac{1}{2}\left[G_n^{(1)}(\rho,\rho',k_\rho)+G_n^{(2)}(\rho,\rho',k_\rho)\right]$$

where

$$G_n^{(1)}(\rho,\rho',k_\rho)=H_n^{(2)}(k_\rho\rho)\,H_n^{(2)}(k_\rho\rho')$$

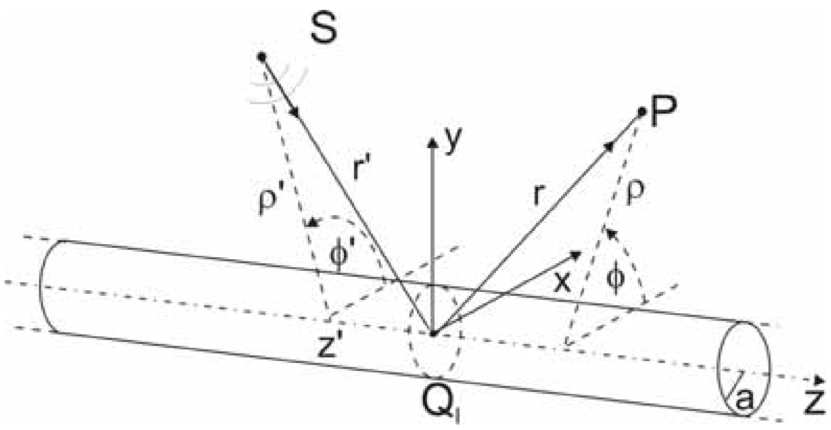
$$\bar{G}_n^{(2)}(\rho,\rho',k_\rho)=\left[\sigma_n^e(k_\rho a)\,H_n^{(2)}(k_\rho\rho)\right]\left[\sigma_n^e(k_\rho^* a)\,H_n^{(2)}(k_\rho\rho')\right]$$

$$\sigma_n^e(k_\rho a)=e^{j\vartheta_n(k_\rho a)} \qquad (\,k_\rho \text{ real})$$

# INCREMENTAL CONTRIBUTION

- Using it in the ITD-FT convolution process leads to

$$\psi_n(Q_l) = \psi_n^{(1)}(Q_l) + \psi_n^{(2)}(Q_l)$$




- by inverse FT

$$\psi_n^{(i)} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_n^{(i)}(k_\rho \rho)\ U_n^{(i)}(k'_\rho \rho') e^{-j(k_z z - k'_z z')} dk_z dk'_z$$

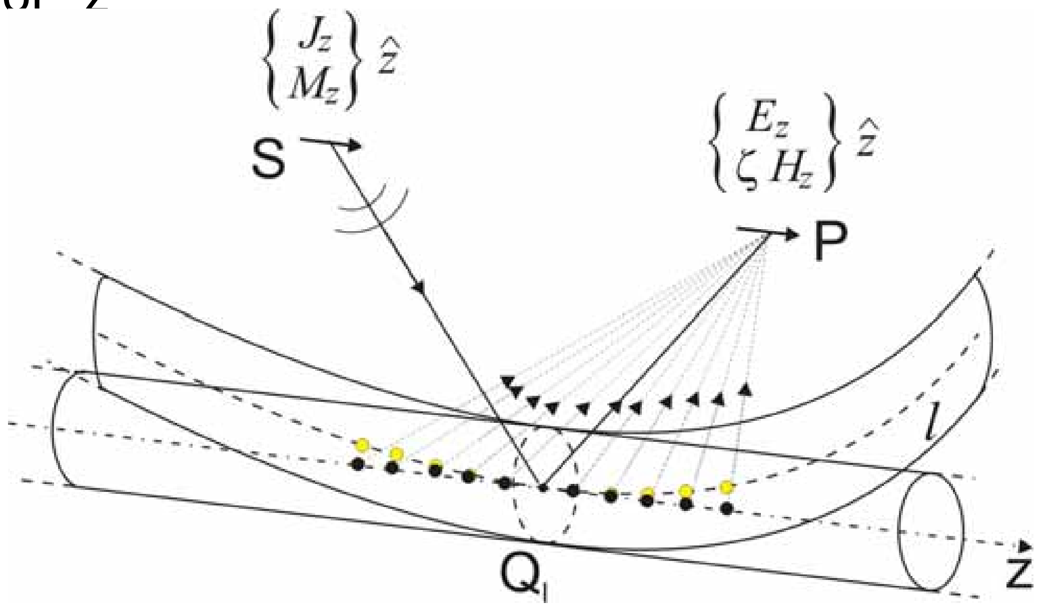
with

$$U_n^{(i)}(k_\rho^{(\prime)} \rho^{(\prime)}) = \begin{Bmatrix} 1 \\ \sigma_n^e(k_\rho^{(\prime)} a) \end{Bmatrix} H_n^{(2)}(k_\rho^{(\prime)} \rho^{(\prime)}) \qquad ; \qquad (i) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

***(Reciprocity)***

# ELECTROMAGNETIC CASE

- In the electromagnetic case for z-directed dipole illumination observation




$\Psi^{(e,h)}$ → EM Vector Potentials

$$\left(E_z, \zeta H_z\right) = \left(k^2 - k_z^2\right) \Psi^{(e,h)}$$

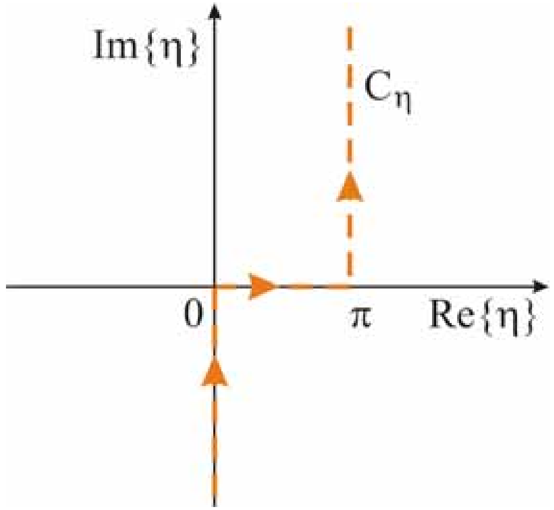
Thus, $\left(d E_z^s, \zeta d H_z^s\right)_n$ is obtained after replacing in $\psi_n^{(i)}$

$U_n^{(i)}(k_\rho^{(\prime)} \rho^{(\prime)})$ by $k_\rho^{(\prime)} U_n^{(i)}(k_\rho^{(\prime)} \rho^{(\prime)})$

# HIGH-FREQUENCY EXPRESSIONS (I)

- In order to find tractable expressions for $\psi_n^{(i)}$ first we use

$$H_n^{(2)}(k_\rho \rho) = \frac{1}{\pi} \int_{C_\alpha} e^{-jk_\rho \rho \cos\alpha} e^{-jn\alpha} e^{-jn\pi/2} d\alpha$$




- Next, collecting terms of the product of the two spectrum functions yields the four fold spectral integral representation for each $\left(dE_z^s, \zeta dH_z^s\right)_n$ as

$$\int_{C_\theta} \int_{C_{\theta'}} \int_{C_\alpha} \int_{C_{\alpha'}} \sigma_n^{e,h}(k_\rho a) \sigma_n^{e,h}(k_\rho^* a) e^{-jn(\alpha+\alpha')} e^{-jk[r(\theta,\alpha)+r'(\theta',\alpha')]} \sin^2\theta \sin^2\theta' \, d\alpha' \, d\alpha \, d\theta' \, d\theta$$

- In the spherical spectral domain $k_z^{(\prime)} = k\cos\theta^{(\prime)}$ , $C_\eta \equiv (-j\infty, \pi + j\infty)$

$$r^{(\prime)}(\theta^{(\prime)}, \alpha^{(\prime)}) = r^{(\prime)}\left(\sin\beta^{(\prime)} \sin\theta^{(\prime)} \cos\alpha^{(\prime)} \, (+/-) \cos\beta^{(\prime)} \cos\theta^{(\prime)}\right)$$

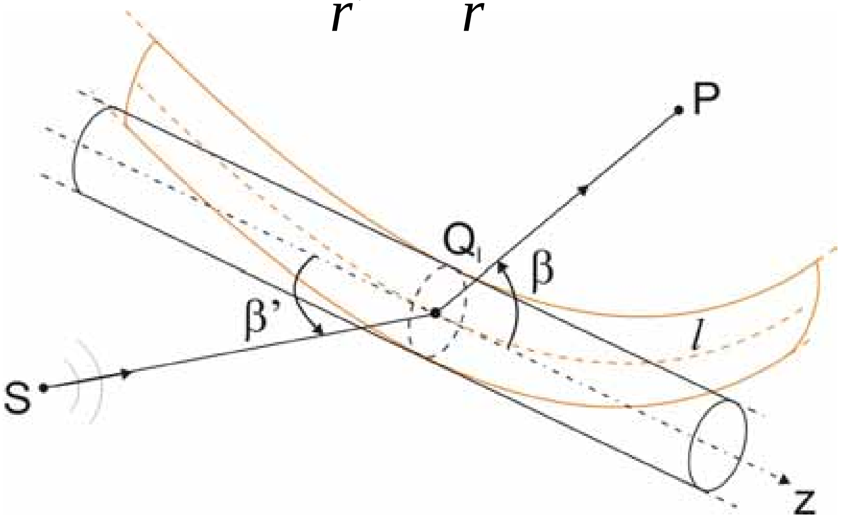
- Standard ITD form.

# HIGH-FREQUENCY EXPRESSIONS (II)

- The incremental electromagnetic scattered field is,

$$\left(dE_z^s, \zeta dH_z^s\right)_n = \left[e^{-jn\pi}\, e^{jn\gamma_s}\, c_n^{(e,h)}(\beta,\beta',a)\right] \sin\beta \sin\beta' \,\frac{e^{-jkr'}}{r'}\,\frac{e^{-jkr}}{r}$$

$$c_n^{(e,h)}(\beta,\beta',a) = \frac{J_n^{(\prime)}\left(ka\sqrt{\sin\beta\sin\beta'}\right)}{H_n^{(\prime)(2)}\left(ka\sqrt{\sin\beta\sin\beta'}\right)}$$




- ITD stationary point

$$\theta = \theta' \quad ; \quad \cos(\alpha+\alpha') = \frac{1-\cos\beta\cos\beta'}{\sin\beta\sin\beta'}$$

$$\sin^2\theta = \sin\beta\sin\beta' \quad ; \quad \gamma_s = \cos^{-1}\frac{1-\cos\beta\cos\beta'}{\sin\beta\sin\beta'}$$

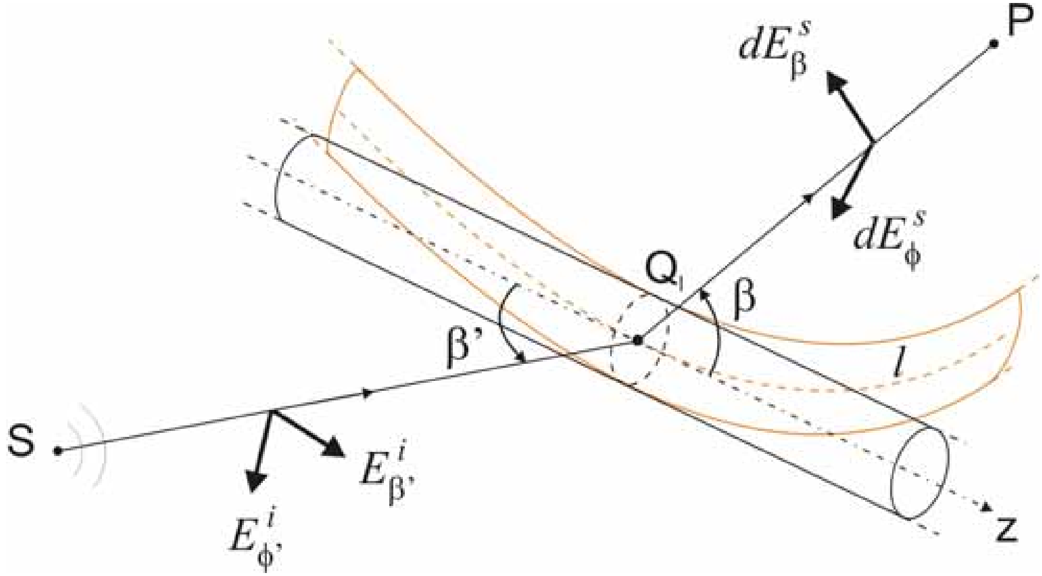
- well-behaved at any incidence and observation aspects

# THE DYADIC SCATTERING COEFFICIENT

$$\begin{pmatrix} dE^s_\beta(P) \\ dE^s_\phi(P) \end{pmatrix} = \begin{pmatrix} \mathcal{D}^s_e(\beta,\beta',\phi,\phi' a) & 0 \\ 0 & \mathcal{D}^s_h(\beta,\beta',\phi,\phi' a) \end{pmatrix} \begin{pmatrix} E^i_{\beta'} \\ E^i_{\phi'} \end{pmatrix} \frac{e^{-jkr}}{4\pi r}$$

$$\mathcal{D}^s_{(e,h)}(\beta,\beta',a) = 4j \sum_{n=-\infty}^{\infty} e^{-jn(\phi-\phi')}\, e^{-jn\pi}\, e^{jn\gamma_s}\, c_n^{(e,h)}(\beta,\beta',a)$$

$$c_n^{(e,h)}(\beta,\beta',a) = \frac{J_n^{(\ ,')}\left(ka\sqrt{\sin\beta \sin\beta'}\right)}{H_n^{(2)(\ ,')}\left(ka\sqrt{\sin\beta \sin\beta'}\right)}$$




- well-behaved at any incidence and observation aspect, including $\beta = 0$ $\beta' = 0$
- The expected transitional behavior of the field is reconstructed by numerical integration of the incremental contributions along the curved axis of the actual cylindrical configuration.

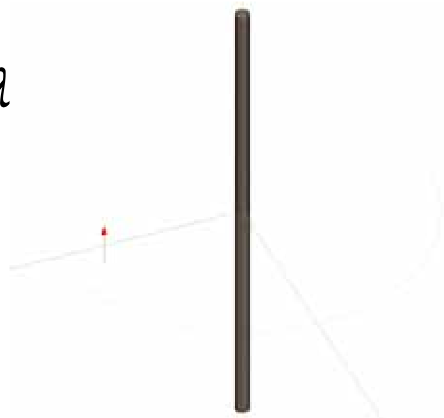
# NUMERICAL RESULTS

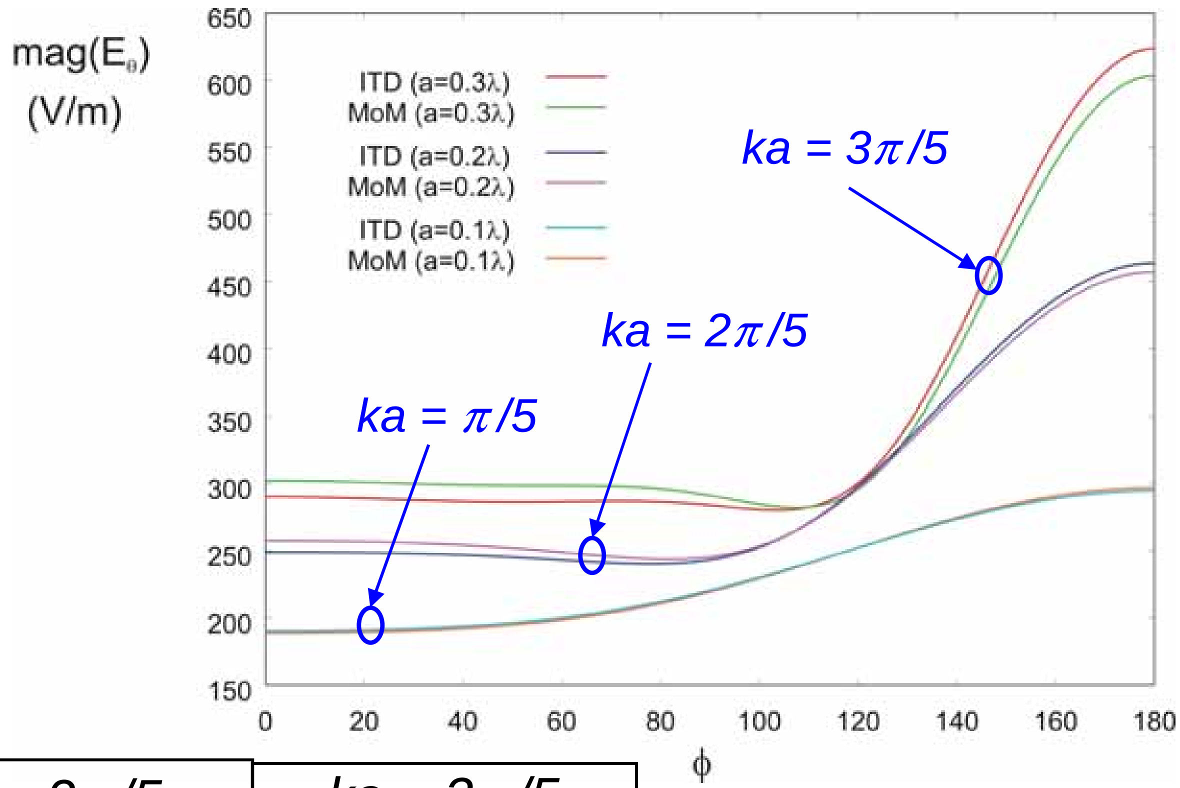
- Results from simulations are compared with a MoM solution (Feko™)
- Straight uniform cylinder – azimuthal scan

*x’* = 6 $\lambda$

*r* = 8 $\lambda$

*L* = 20 $\lambda$






## CPU times

| | $ka = \pi/5$ | $ka = 2\pi/5$ | $ka = 3\pi/5$ |
|---|---|---|---|
| **ITD** | 1.1 s ($n_{av}$ = 3.6) | 2.3 s ($n_{av}$ = 3.8) | 4 s ($n_{av}$ = 4) |
| **MoM** | 15.3 s | 148.4 s | 279.8 s |

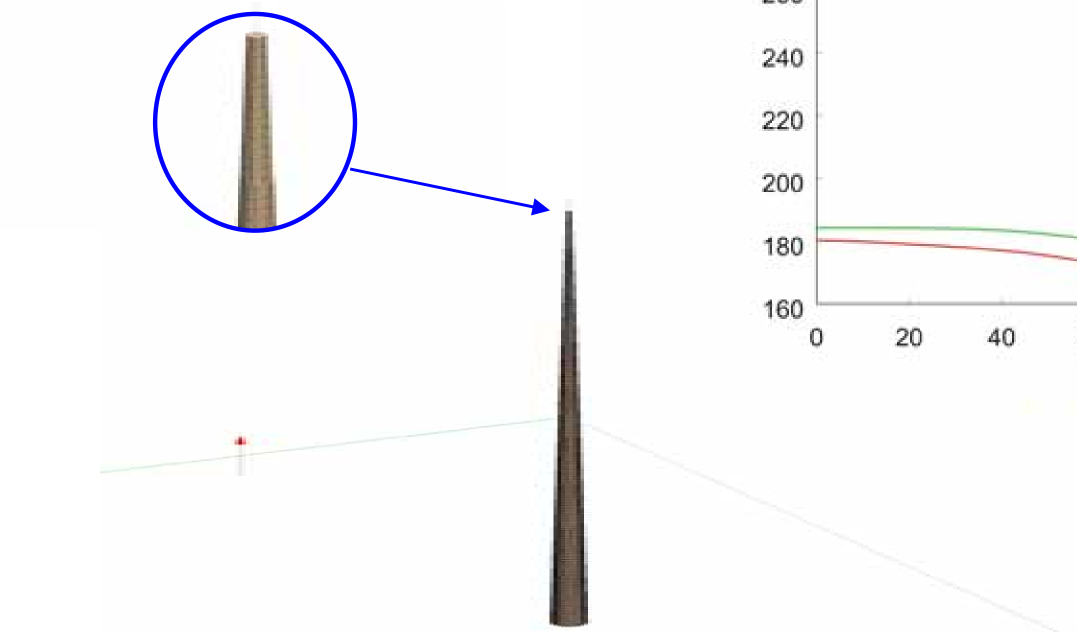
# NUMERICAL RESULTS

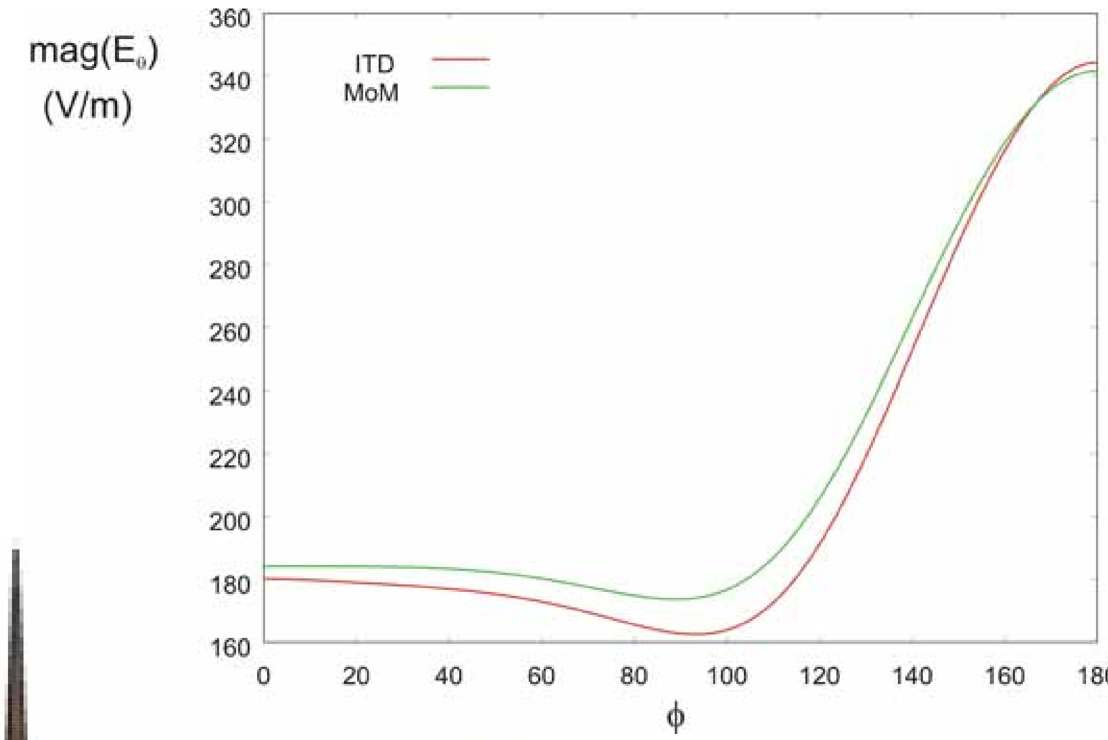
- Results from simulations are compared with a MoM solution (Feko™)
- Straight tapered cylinder – azimuthal scan

*x’ = 8 λ*

*r = 10 λ*

*L = 20 λ*






$a_{min} = 0.05\ \lambda$

$a_{max} = 0.4\ \lambda$

**CPU times**

| | |
|---|---|
| **ITD** | 1.2 s ($n_{av}$ = 5.5) |
| **MoM** | 17 min 6 s |

# NUMERICAL RESULTS

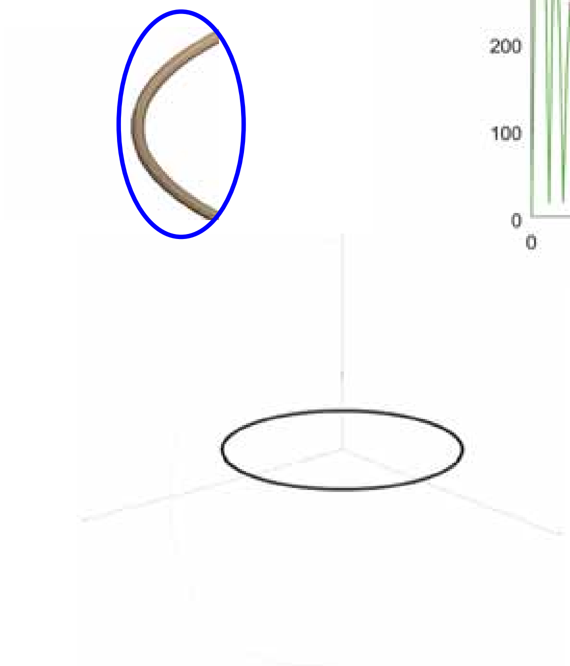
- Results from simulations are compared with a MoM solution (Feko™)
- Circular torus – elevation scan

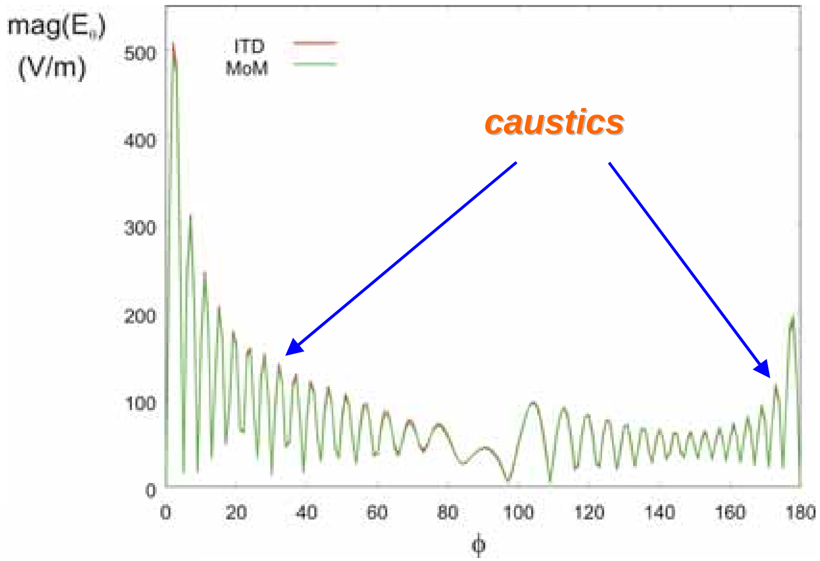
*a = 0.1 λ*

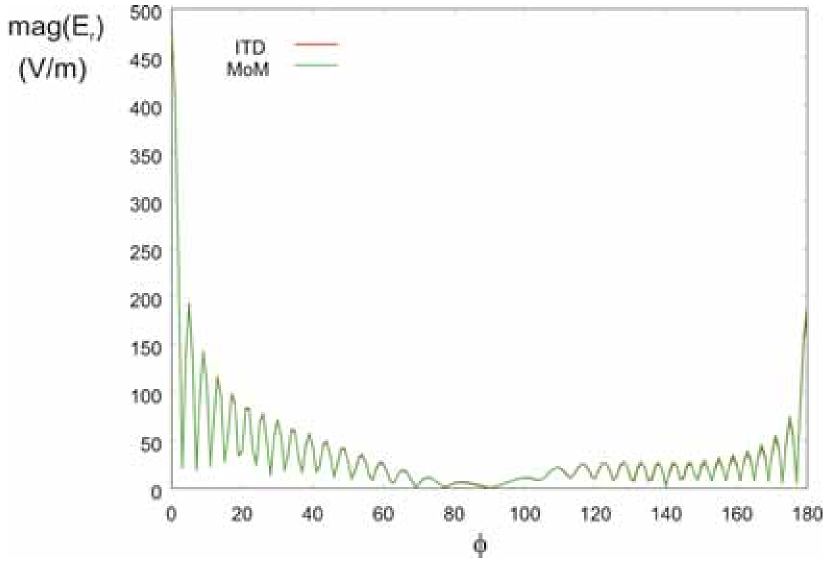
$R_T = 8\,\lambda$

*z' = 5 λ*

*r = 15 λ*





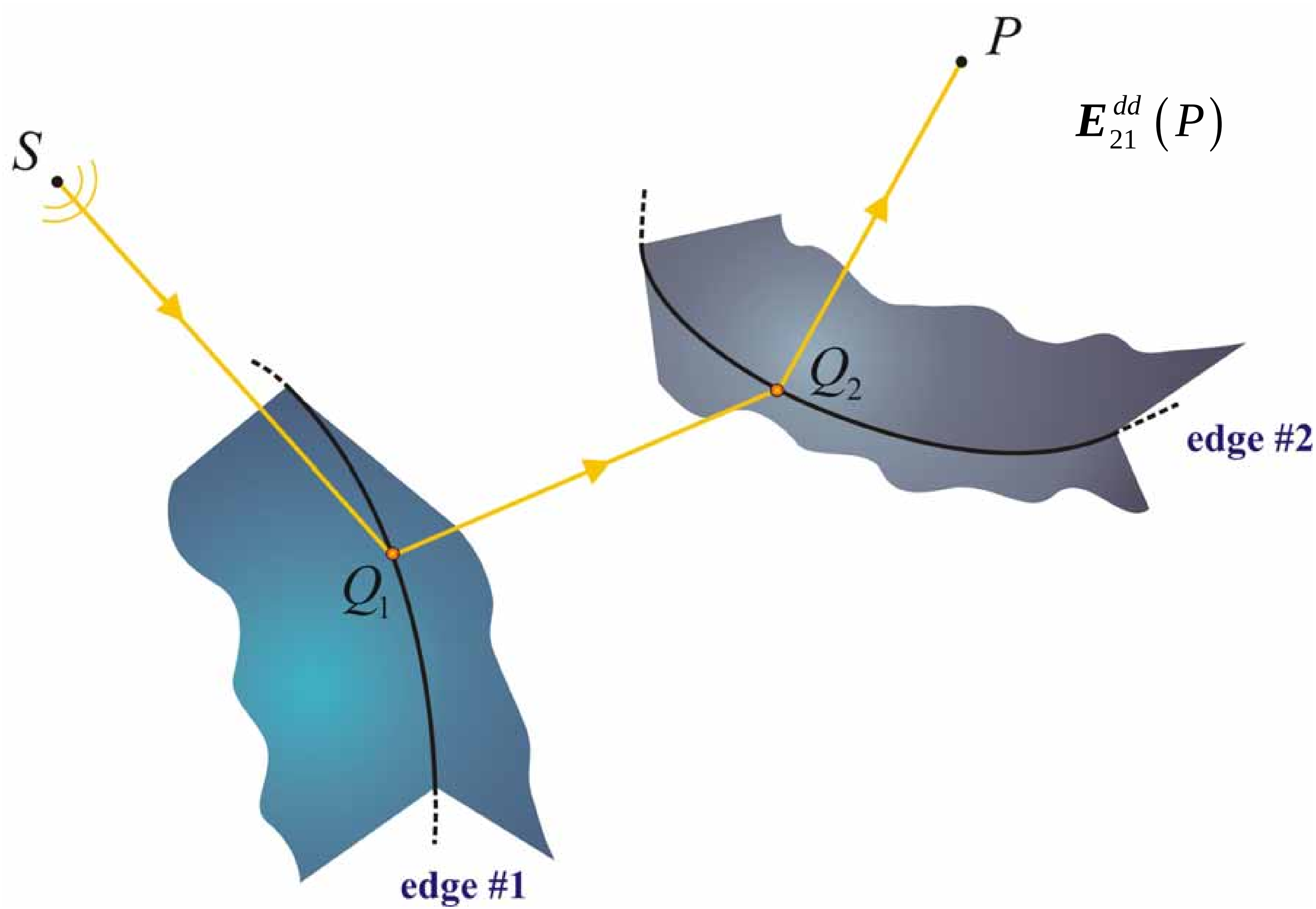

**CPU times**

**ITD** 2.2 s ($n_{av}$ = 3.7)

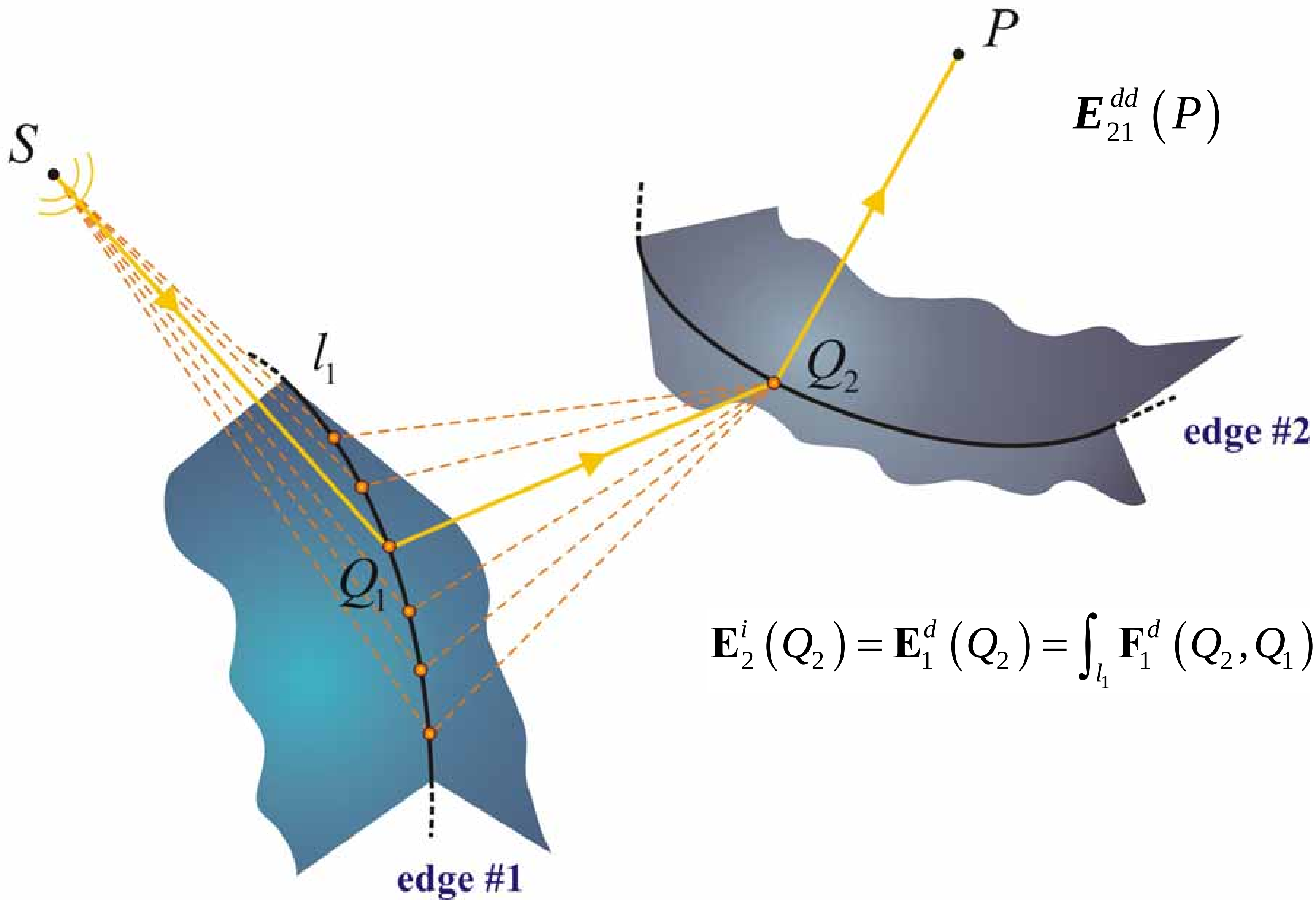
**MoM** 2 min 25 s

# Double Edge Diffraction

# ITD FORMULATION FOR DOUBLE EDGE DIFFRACTION

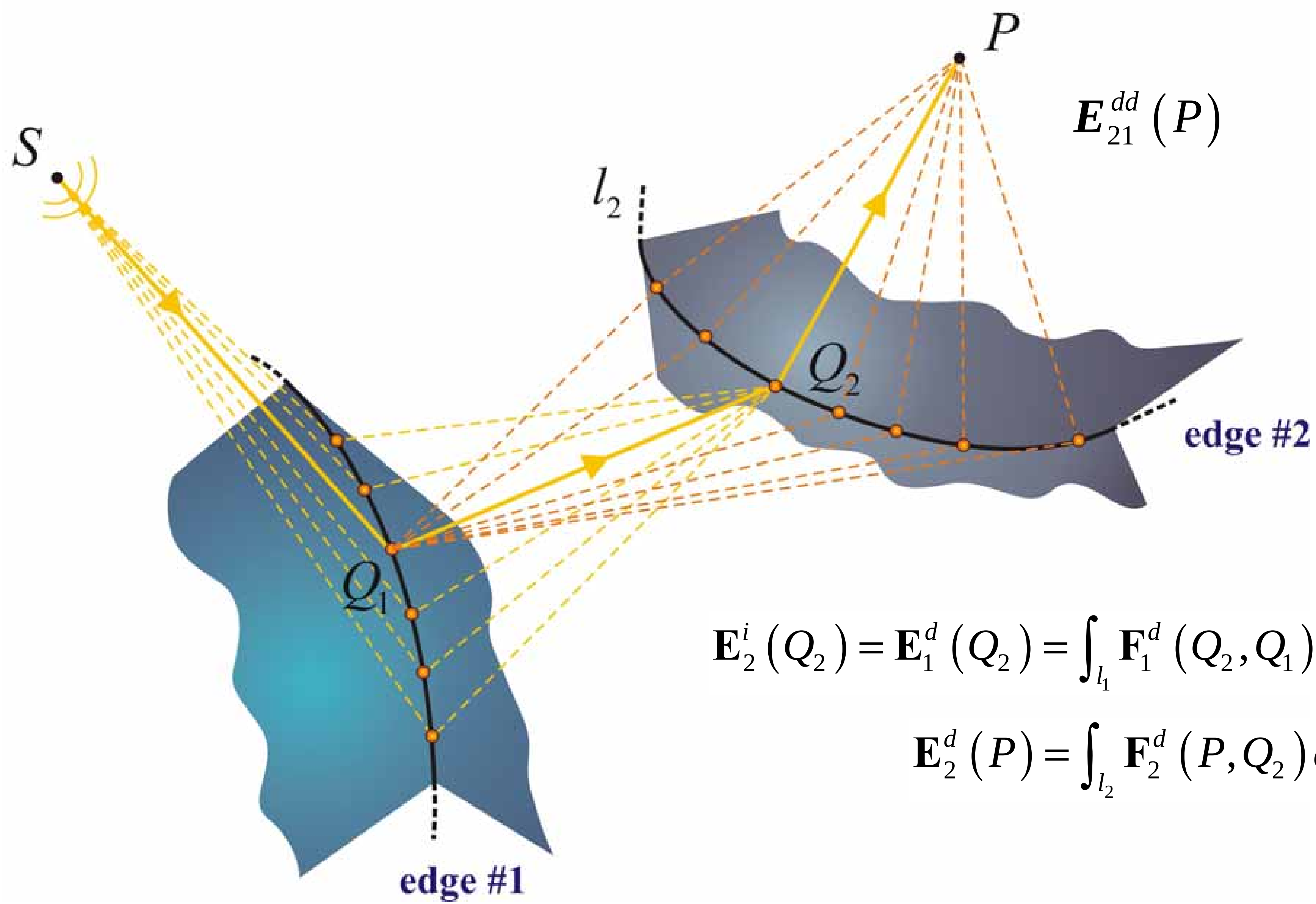


# ITD FORMULATION FOR DOUBLE EDGE DIFFRACTION




$$\mathbf{E}_2^i(Q_2) = \mathbf{E}_1^d(Q_2) = \int_{l_1} \mathbf{F}_1^d(Q_2, Q_1)\, dl_1$$

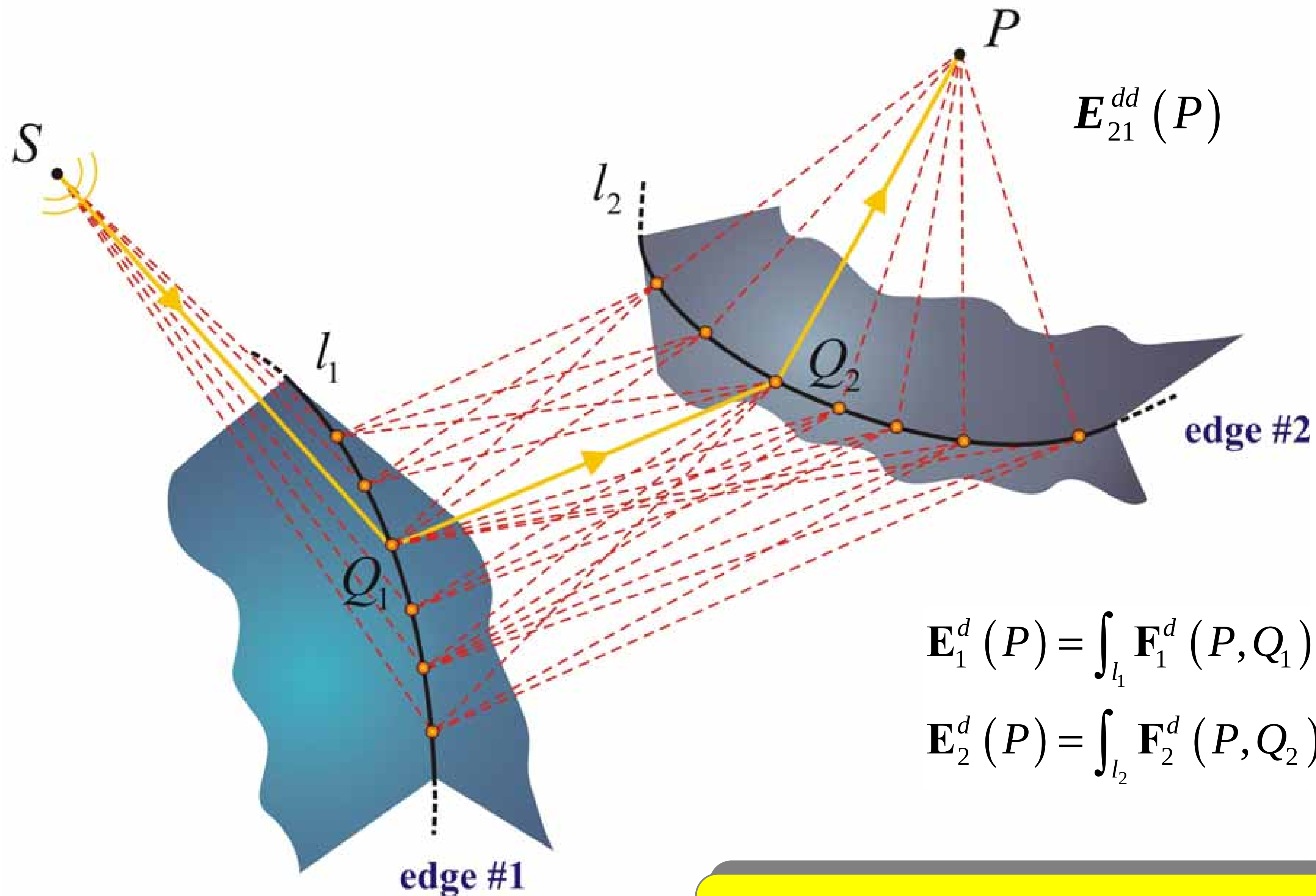
# ITD FORMULATION FOR DOUBLE EDGE DIFFRACTION




$$\mathbf{E}_2^i(Q_2) = \mathbf{E}_1^d(Q_2) = \int_{l_1} \mathbf{F}_1^d(Q_2, Q_1)\, dl_1$$

$$\mathbf{E}_2^d(P) = \int_{l_2} \mathbf{F}_2^d(P, Q_2)\, dl_2$$

# ITD FORMULATION FOR DOUBLE EDGE DIFFRACTION

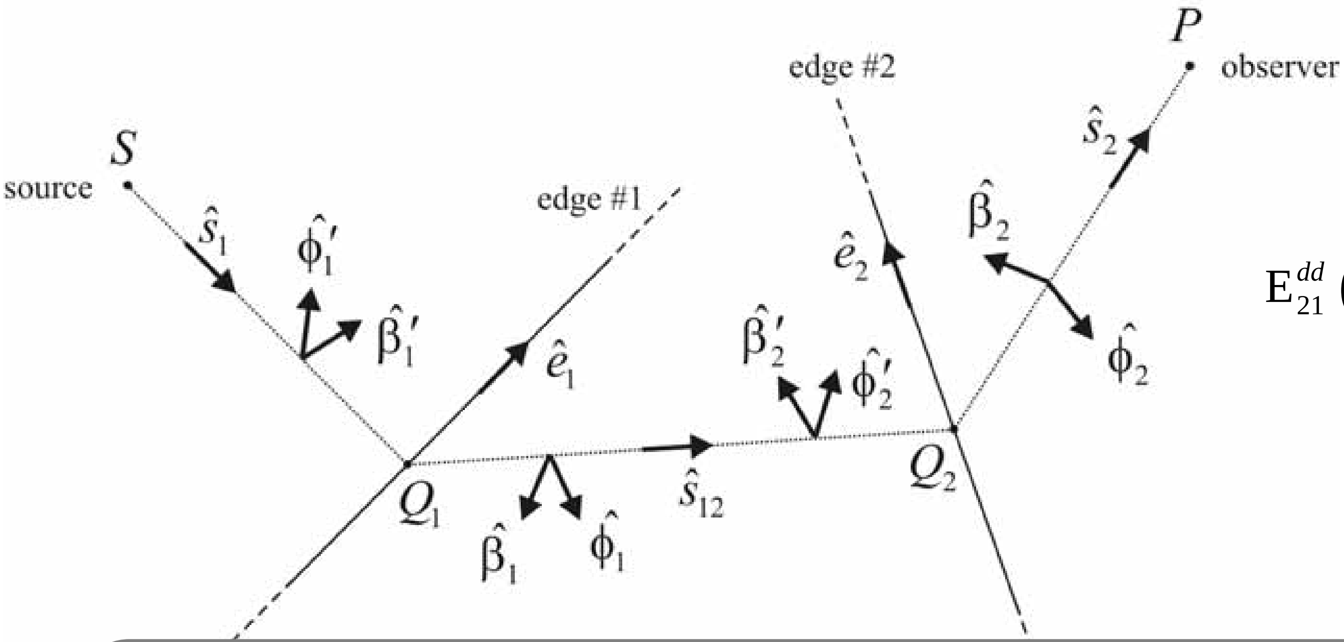



$$\mathbf{E}_1^d(P) = \int_{l_1} \mathbf{F}_1^d(P, Q_1)\, dl_1$$

$$\mathbf{E}_2^d(P) = \int_{l_2} \mathbf{F}_2^d(P, Q_2)\, dl_2$$

$$\boxed{\mathbf{E}_{21}^{dd}(P) = \int_{l_2}\int_{l_1} \mathbf{F}_{21}^{dd}(P, Q_2, Q_1)\, dl_1 dl_2}$$

# ITD FORMULATION FOR DOUBLE EDGE DIFFRACTION




$$\mathrm{E}_{21}^{dd}(P) = \int_{l_2}\int_{l_1} \mathbf{F}_{21}^{dd}(P, Q_2, Q_1)\, dl_1 dl_2$$

$$\begin{pmatrix} F_{\beta_2}^{dd}(P, Q_2, Q_1) \\ F_{\phi_2}^{dd}(P, Q_2, Q_1) \end{pmatrix} = \underline{\underline{\mathcal{D}_2}}(\upsilon_2, \phi_2, \phi_2') \cdot \underline{\underline{\mathcal{M}}}(\gamma_{12}) \cdot \underline{\underline{\mathcal{D}_1}}(\upsilon_1, \phi_1, \phi_1') \cdot \begin{pmatrix} E_{\beta_1'}^{i}(Q_1) \\ E_{\phi_1'}^{i}(Q_1) \end{pmatrix} \frac{e^{-jks_{12}}}{2\pi s_{12}} \frac{e^{-jks_2}}{2\pi s_2}$$

$$\begin{pmatrix} E_{\beta_2'}^{i} \\ E_{\phi_2'}^{i} \end{pmatrix} = \underline{\underline{\mathcal{M}}}(\gamma_{12}) \cdot \begin{pmatrix} E_{\beta_1}^{d} \\ E_{\phi_1}^{d} \end{pmatrix} = \begin{pmatrix} \cos\gamma_{12} & -\sin\gamma_{12} \\ \sin\gamma_{12} & \cos\gamma_{12} \end{pmatrix} \cdot \begin{pmatrix} E_{\beta_1}^{d} \\ E_{\phi_1}^{d} \end{pmatrix}$$