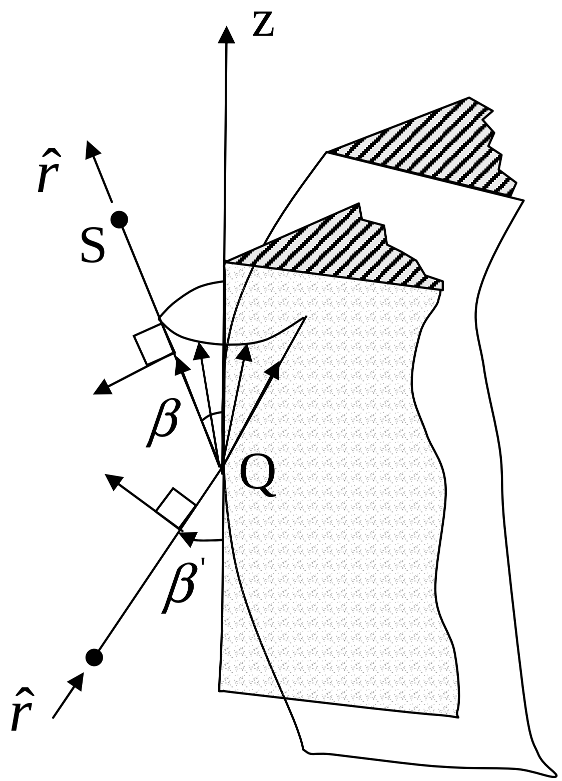
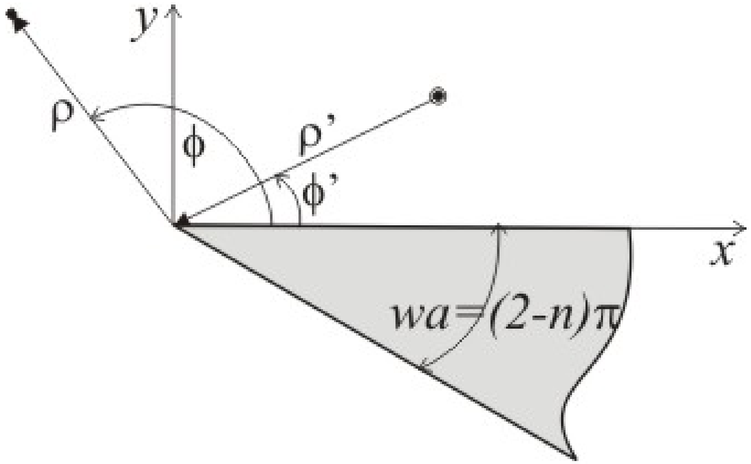
Scattering from a perfectly conducting half-plane

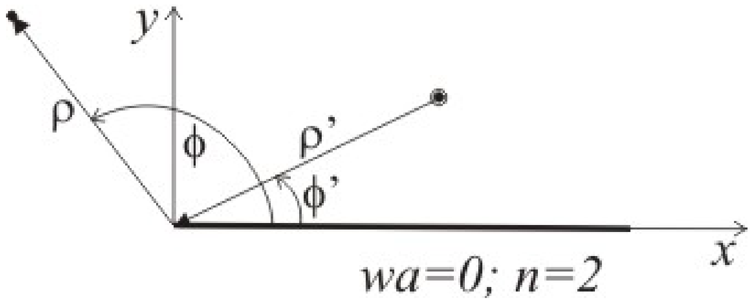
CANONICAL PROBLEMS



Wedge problem



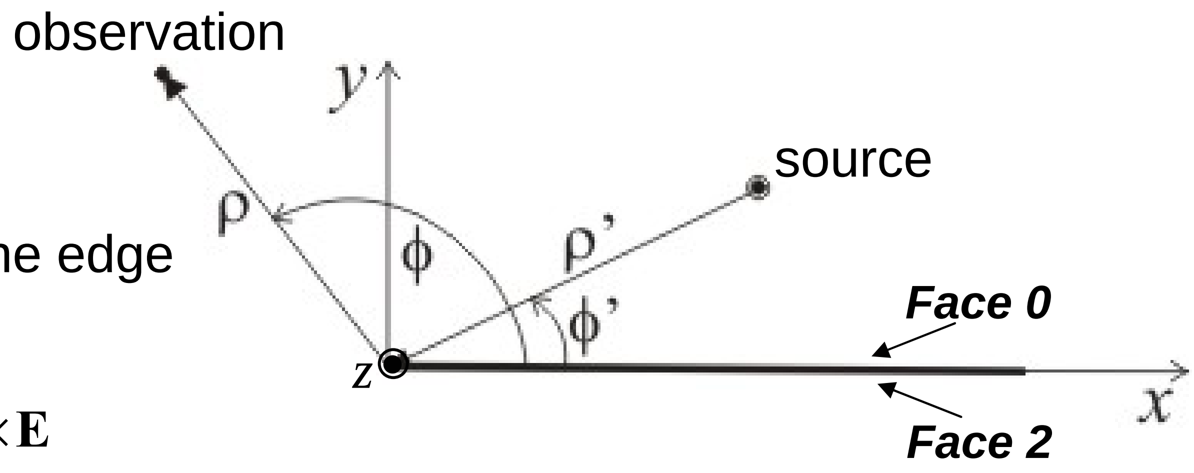
Half plane problem



RADIATION FROM THE HALF PLANE

Assumptions:

1. normal incidence
2. electric line source
3. source far from the vertex of the edge



$$E_z = -\frac{\omega\mu I_e}{4} G; \qquad H_z = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}$$

Green's function

$$G = \sqrt{\frac{2j}{\pi k \rho'}} e^{-jk\rho'} F(k\rho)$$

Modal solution

$$F(k\rho) = \frac{1}{2} \sum_{m=0}^{\infty} \varepsilon_m J_{m/2}(k\rho) e^{j(m/2)(\pi/2)} \left[\frac{m}{2} \cos\Phi^- \pm \frac{m}{2} \cos\Phi^+ \right]; \qquad \varepsilon_m = \begin{cases} 0 & m = 0 \\ 1 \end{cases} \qquad \Phi^{\mp} = (\phi \mp \phi')$$

hard ($+$) / *soft* ($-$)

Rapidly convergent for $k\rho < 1$ (about 15 terms)
for $1 < k\rho < 10$ (about 40 terms)
Slowly convergent for $k\rho > 10$

High frequency asymptotic expansion

HIGH FREQUENCY ASYMPTOTIC EXPANSION

Converting the infinite series into an integral

$$F(k\rho) = \int_{C_\alpha} S(\alpha) e^{k\rho f(\alpha)} d\alpha$$

deformation onto SDP

finding saddle points and poles

How to do:

1. Expansion of cosine terms into complex exponentials

2. Integral expression for Bessel functions

3. Exchanging the order of summation and integration

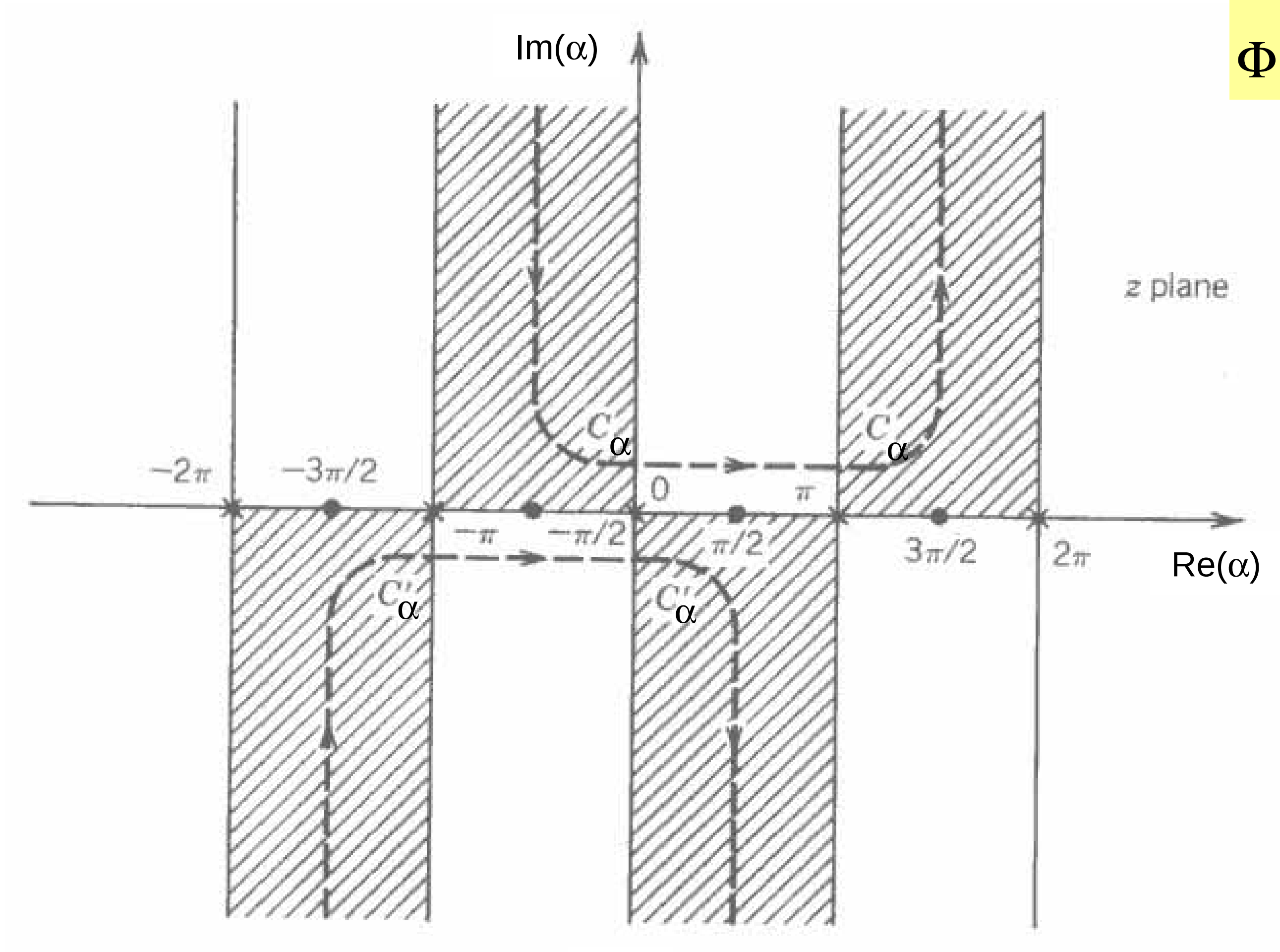
HIGH FREQUENCY ASYMPTOTIC EXPANSION

$$F(k\rho) = F_1(k\rho) \pm F_2(k\rho)$$

$$F_i(k\rho) = \frac{1}{8\pi j} \int_{C_{\alpha'} - C_\alpha} S_i(\alpha) e^{k\rho f_i(\alpha)} d\alpha \quad \text{where} \begin{cases} S_1(\alpha) = \cot\left(\dfrac{\Phi^- + \alpha}{4}\right) \\ S_2(\alpha) = \cot\left(\dfrac{\Phi^+ + \alpha}{4}\right) \end{cases} \text{and } f_1(\alpha) = f_2(\alpha) = j\cos\alpha$$

$$\Phi^- = \phi - \phi'$$

$$\Phi^+ = \phi + \phi'$$

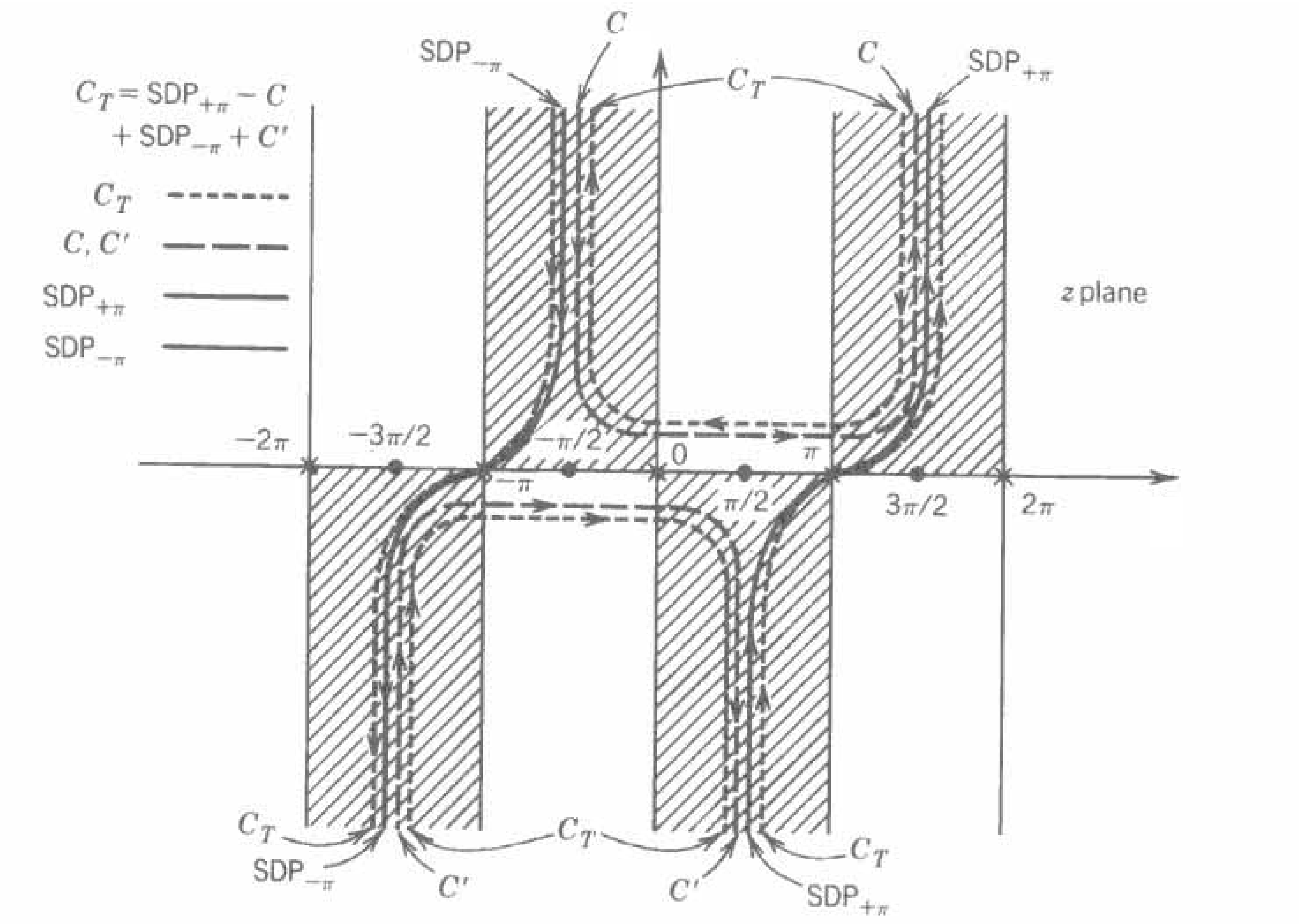


HIGH FREQUENCY ASYMPTOTIC EXPANSION

1. Choice of closed contour $C_T = C_{\alpha'} + SDP_{+\pi} - C_\alpha + SDP_{-\pi}$

$$F_1(k\rho) = \frac{1}{8\pi j} \int_{C_{\alpha'} - C_\alpha} S_1(\alpha) e^{k\rho f_1(\alpha)} d\alpha =$$

$$= \frac{1}{8\pi j} \oint_{C_T} S_1(\alpha) e^{k\rho f_1(\alpha)} d\alpha - \frac{1}{8\pi j} \int_{SDP_{+\pi}} S_1(\alpha) e^{k\rho f_1(\alpha)} d\alpha - \frac{1}{8\pi j} \int_{SDP_{-\pi}} S_1(\alpha) e^{k\rho f_1(\alpha)} d\alpha$$



RESIDUES CONTRIBUTION

2. From $\oint$ = residues of the poles inside C_T

$$\frac{1}{8\pi j}\oint_{C_T} S_1(\alpha)e^{k\rho f_1(\alpha)}d\alpha = 2\pi j\sum_{i=1}^{Np}\operatorname{Res}(\alpha_i)U(\alpha_i)$$

3. Poles of $S_1(\alpha) \longrightarrow \cot\left(\frac{\Phi^- + \alpha}{4}\right) \qquad \Rightarrow \left(\frac{\Phi^- + \alpha}{4}\right) = \pi N \qquad N = 0, \pm 1, \pm 2, \ldots$

Easy calculation

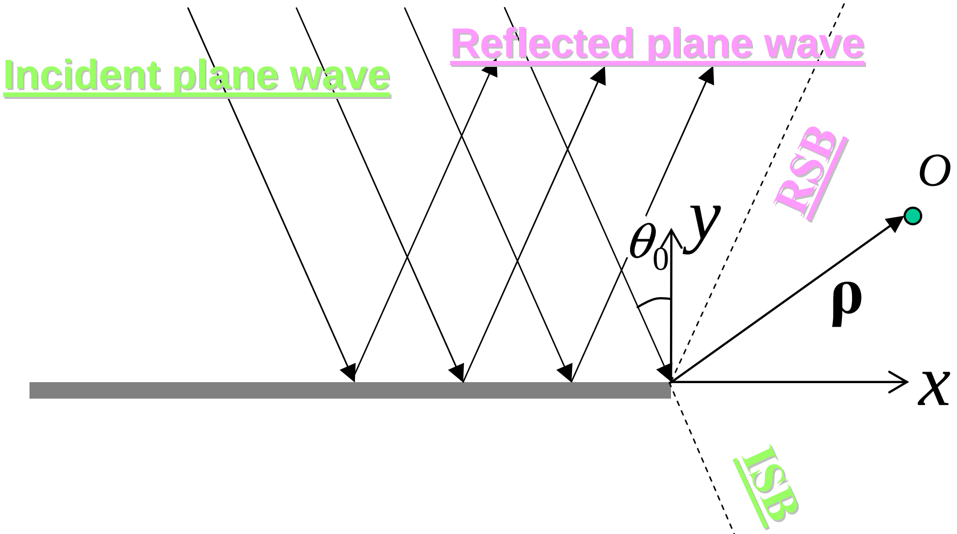
$\alpha = -\Phi^- + 4\pi N$ real poles

Incident GO

$$2\pi j\sum_{i=1}^{Np}\operatorname{Res}(\alpha_i)U(\alpha_i) = e^{jk\rho\Phi^-}U(\pi - \Phi^-)$$

Reflected GO

$$2\pi j\sum_{i=1}^{Np}\operatorname{Res}(\alpha_i)U(\alpha_i) = e^{jk\rho\Phi^+}U(\pi - \Phi^+)$$



SDP CONTRIBUTION

4. Non uniform asymptotic approximation (isolated poles – saddle points)

$$\frac{1}{8\pi j}\int_{SDP_{+\pi}} S_1(\alpha)e^{-jk\rho\cos\alpha}d\alpha \quad + \quad \frac{1}{8\pi j}\int_{SDP_{-\pi}} S_1(\alpha)e^{-jk\rho\cos\alpha}d\alpha$$

$$A\int_{SDP} f(z)e^{\Omega q(z)}d\alpha \approx A\sqrt{\frac{2\pi}{\Omega q''(z_s)}}f(z_s)e^{\Omega q(z_s)}$$

Asymptotic evaluation parameters

integrand	expansion argument	phase function	saddle point
$f(z)=S_1(\alpha)$	$\Omega=k\rho$	$q(z)=-j\cos\alpha$	$q'(z)=j\sin\alpha=0$ $z_s=\alpha_s=\pm\pi$

$\alpha_s=+\pi$ $\qquad\qquad$ $\alpha_s=-\pi$

$$\frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\cot\left(\frac{\pi+\Phi^-}{4}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}} \quad + \quad \frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\cot\left(\frac{\pi-\Phi^-}{4}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}$$

$$\boxed{F_{1\,SDP}(k\rho)\approx\frac{e^{-j\pi/4}}{2\sqrt{2\pi}}\frac{1}{\cos\left(\frac{\Phi^-}{2}\right)}\frac{e^{-jk\rho}}{\sqrt{k\rho}}}$$

SDP CONTRIBUTION

Equivalently…

$$\frac{1}{8\pi j}\int_{SDP_{+\pi}} S_2(\alpha)e^{-jk\rho\cos\alpha}d\alpha \quad + \quad \frac{1}{8\pi j}\int_{SDP_{-\pi}} S_2(\alpha)e^{-jk\rho\cos\alpha}d\alpha$$

$$F_{2\,SDP}(k\rho)=\frac{e^{-j\pi/4}}{2\sqrt{2\pi}}\frac{1}{\cos\left(\frac{\Phi^{+}}{2}\right)}\frac{e^{-jk\rho}}{\sqrt{k\rho}}$$

<u>Summary of the non-uniform first order asymptotic evaluation</u>

$$F(k\rho)=\frac{-e^{-j\pi/4}}{2\sqrt{2\pi}}\left[\frac{1}{\cos\left(\frac{\Phi^{-}}{2}\right)}\pm\frac{1}{\cos\left(\frac{\Phi^{+}}{2}\right)}\right]\frac{e^{-jk\rho}}{\sqrt{k\rho}}+e^{jk\rho\Phi^{-}}U\left(\pi-\Phi^{-}\right)\pm e^{jk\rho\Phi+}U\left(\pi-\Phi^{+}\right)$$

SDP CONTRIBUTION

5. Uniform asymptotic approximation (poles near saddle points)

$$\frac{1}{8\pi j}\int_{SDP_{+\pi}} S_1(\alpha)e^{-jk\rho\cos\alpha}d\alpha \quad + \quad \frac{1}{8\pi j}\int_{SDP_{-\pi}} S_1(\alpha)e^{-jk\rho\cos\alpha}d\alpha$$

6. Pauli – Clemmow modified method of steepest descent path

6a. Expansion of the phase term $f(\alpha)=f(\alpha_s)-\mu^2$

6b. Expansion of the integrand $F(\mu)=S_1(\alpha)[f(\alpha)-f(\alpha_i)]\dfrac{d\alpha}{d\mu}=\sum_{m=0}^{\infty}c_m\mu^m$

6c. Assuming $f(\alpha)-f(\alpha_i)=f(\alpha_s)-f(\alpha_i)-\mu^2=-(\mu^2+ja);\qquad a=j[f(\alpha_s)-f(\alpha_i)]$

6d. Change of integration variable

$$\int_{-\infty}^{\infty}\frac{e^{-k\rho\mu^2}}{\mu^2+ja}d\mu=2e^{jk\rho a}\sqrt{\frac{\pi}{a}}\int_{\sqrt{k\rho a}}^{\infty}e^{-j\tau^2}d\tau \longrightarrow F[k\rho a]$$

Fresnel transition function

SDP CONTRIBUTION

Uniform evaluation

$$\frac{1}{8\pi j}\int_{SDP_{\pm\pi}} S_1(\alpha)e^{-jk\rho\cos\alpha}d\alpha = \frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\cot\left(\frac{\pi\pm\Phi^-}{4}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^{\pm}\left(\Phi^-\right)\right]$$

$$\frac{1}{8\pi j}\int_{SDP_{\pm\pi}} S_2(\alpha)e^{-jk\rho\cos\alpha}d\alpha = \frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\cot\left(\frac{\pi\pm\Phi^+}{4}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^{\pm}\left(\Phi^+\right)\right]$$

$$F\left[k\rho g^{\pm}(x)\right] = j\underbrace{\left[f(\alpha_s) - f(\alpha_i)\right]}_{\boxed{\text{Distance between saddle points and poles (angular separation)}}} = 2j\,\left|\sqrt{k\rho g^{\pm}(x)}\right|\int_{\sqrt{k\rho g^{\pm}(x)}}^{\infty} e^{-j\tau^2}d\tau$$

$$g^{\pm}(x) = 1 + \cos\left[x - 4\pi N^{\pm}\right] \qquad N^{\pm}\text{integer}: \begin{cases} 4\pi N^{+} - x = \pi \\ 4\pi N^{-} - x = -\pi \end{cases}$$

TOTAL RADIATED FIELD

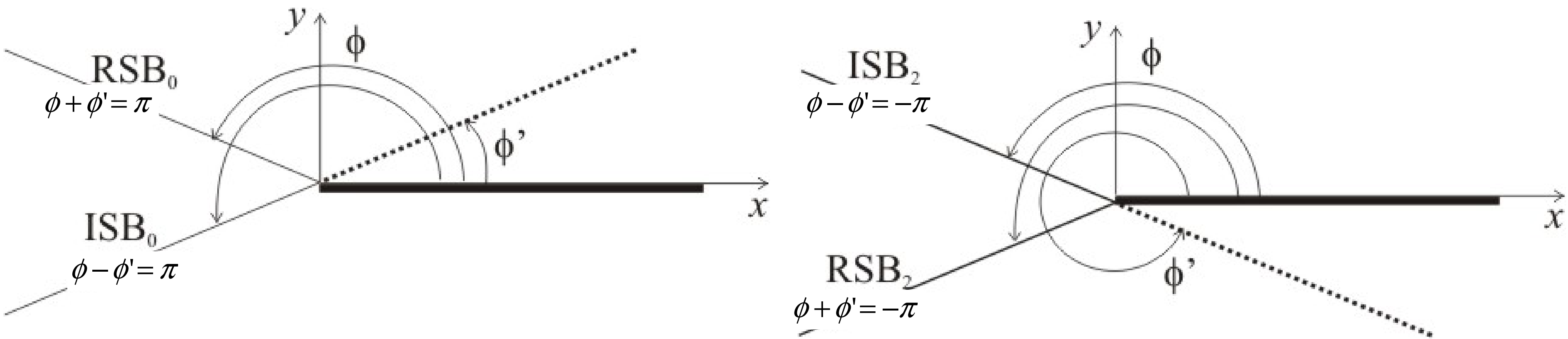
GEOMETRICAL OPTICS FIELDS

Incident GO　*Reflected GO*

$$F_{GO}(k\rho)=\begin{cases} e^{jk\rho\cos(\phi-\phi')}\pm e^{jk\rho\cos(\phi+\phi')} & 0<\phi<\pi-\phi' \\ e^{jk\rho\cos(\phi-\phi')} & \pi-\phi'<\phi<\pi+\phi' \\ 0 & \pi+\phi'<\phi<2\pi \end{cases}$$

$\pi-\phi' \rightarrow$ **ISB$_0$**　$\phi-\phi'=\pi$

$\pi+\phi' \rightarrow$ **RSB$_0$**　$\phi+\phi'=\pi$



TOTAL RADIATED FIELD

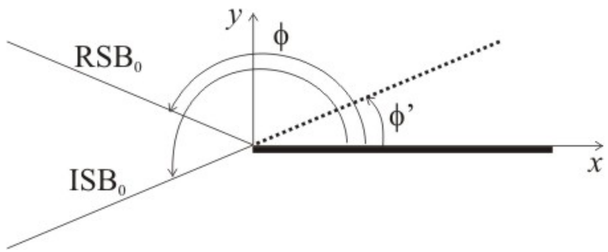
DIFFRACTED FIELDS

$$F_D(k\rho)=-\frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\overset{\text{ISB face 0}}{\boxed{\cot\left(\frac{\pi-\Phi^-}{4}\right)}}\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^-\left(\Phi^-\right)\right]\mp\frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\overset{\text{RSB face 0}}{\boxed{\cot\left(\frac{\pi-\Phi^+}{4}\right)}}\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^-\left(\Phi^+\right)\right]-$$
$$-\frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\underset{\text{ISB face 2}}{\boxed{\cot\left(\frac{\pi+\Phi^-}{4}\right)}}\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^+\left(\Phi^-\right)\right]\mp\frac{e^{-j\pi/4}}{4\sqrt{2\pi}}\underset{\text{RSB face 2}}{\boxed{\cot\left(\frac{\pi+\Phi^+}{4}\right)}}\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^+\left(\Phi^+\right)\right]$$

GENERALIZATION TO A PERFECTLY CONDUCTING WEDGE

$$F_D(k\rho)=-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi}}\cot\left(\frac{\pi-\Phi^-}{2n}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^-\left(\Phi^-\right)\right]\mp\frac{e^{-j\pi/4}}{2n\sqrt{2\pi}}\cot\left(\frac{\pi-\Phi^+}{2n}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^-\left(\Phi^+\right)\right]-$$
$$-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi}}\cot\left(\frac{\pi+\Phi^-}{2n}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^+\left(\Phi^-\right)\right]\mp\frac{e^{-j\pi/4}}{2n\sqrt{2\pi}}\cot\left(\frac{\pi+\Phi^+}{2n}\right)\frac{e^{-jk\rho}}{\sqrt{k\rho}}F\left[k\rho g^+\left(\Phi^+\right)\right]$$

TOTAL RADIATED FIELD



$$\phi' = 30^\circ$$
$$RSB_0 = 150^\circ$$
$$ISB_0 = 210^\circ$$