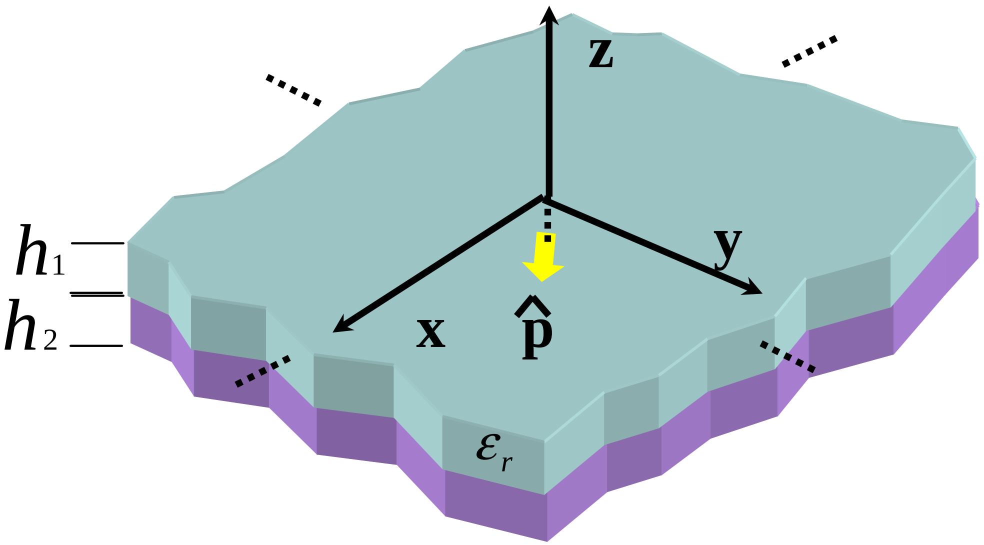
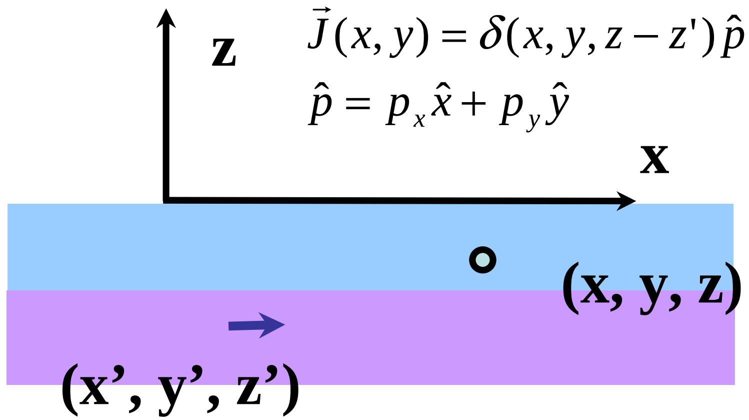
Green’s function of the dielectric slab

SINGLE ELEMENT GF: GEOMETRY





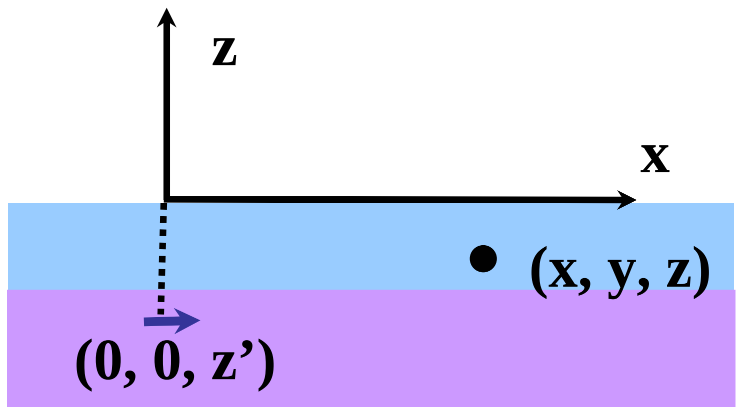
Observation point ➡ $\underline{r} = \underline{\rho} + \hat{z}z \qquad \underline{\rho} = \hat{x}x + \hat{y}y$

Source position ➡ $\underline{r}' = \underline{\rho}' + \hat{z}z' \qquad \underline{\rho}' = \hat{x}x' + \hat{y}y'$

$$\vec{E}(x, y, z; x', y', z')$$

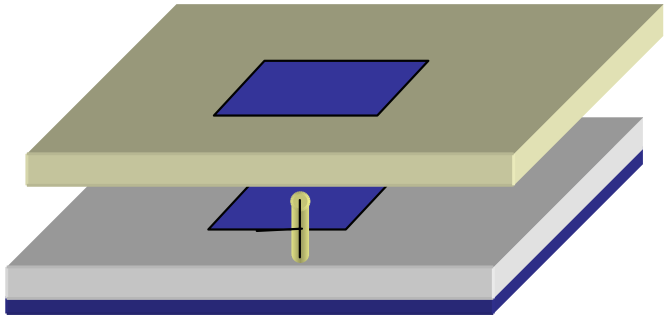
⬇

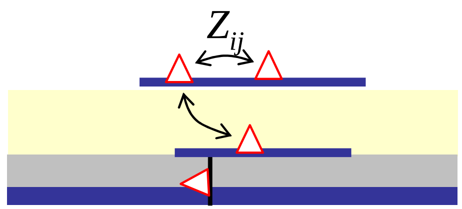
$$\vec{E}(x - x', y - y', z, z')$$



SINGLE ELEMENT GF: APPLICATION

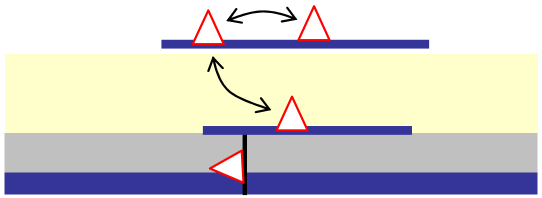
Green’s function for the full-wave analysis of planar structures





• **Spectral domain Method of Moment (MoM)**

Electric Field Integral Equation (EFIE)



$\underline{\underline{\mathbf{Z}}}$ Matrix

$$Z_{ij} \Rightarrow \langle \tilde{\mathbf{T}}^*, \underline{\underline{\tilde{G}}}^{EJ} \tilde{\mathbf{J}} \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\mathbf{T}}^*(k_x, k_y) \cdot \underline{\underline{\tilde{G}}}^{EJ}(k_x, k_y) \cdot \tilde{\mathbf{J}}(k_x, k_y)\, dk_x\, dk_y$$

Spectral dyadic Green’s function of the electric field due to an electric source **J**

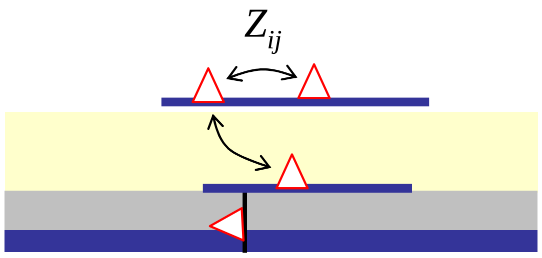
$$\underline{\underline{\tilde{G}}}^{EJ} = -\hat{\alpha}\,\hat{\alpha}\, v_{TE} - \hat{k}_\rho \hat{k}_\rho v_{TM} + \zeta \frac{k_\rho}{k} i_{TM}\, \hat{z}\hat{k}_\rho$$

<u>Directly in the spectral domain from the z-transmission line</u>

SINGLE ELEMENT GF: APPLICATION

- **Spatial domain Method of Moment (MoM)**

  Mixed Potential Integral Equation (MPIE)



$\underline{\underline{\mathbf{Z}}}$ Matrix

$$Z_{ij} \Rightarrow \langle \mathbf{T}, \mathbf{E}_s \rangle = -j\omega \left[\langle \mathbf{T}, \underline{\underline{\mathbf{G}}}_A^{'} * \mathbf{J} \rangle + \langle q_t, K_\phi * q \rangle \right]$$

Spatial Green's functions of the vector and scalar potentials

$$\mathbf{A}'(\mathbf{r}) = \underline{\underline{\mathbf{G}}}_A^{'} * \mathbf{J}, \quad \Phi(\mathbf{r}) = K_\phi * q \qquad \underline{\underline{\mathbf{G}}}_A^{'} = \begin{bmatrix} g_{xx} & 0 & 0 \\ 0 & g_{xx} & 0 \\ g_{xz} & g_{yz} & 0 \end{bmatrix}$$ (for planar source)

Unknown in the spatial domain ⇨ Fourier double integral

$$g(x, y; z, z') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}(k_z; z, z') e^{-j(k_x x + k_y y)} dk_x dk_y \qquad k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

SOMMERFELD INTEGRAL

$$g(x,y;z,z')=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{G}(k_z;z,z')\,e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

k_x, k_y complex plane

$$k_x = k_\rho \cos\alpha \qquad x=\rho\cos\phi$$

$$k_y = k_\rho \sin\alpha \qquad y=\rho\sin\phi$$

$$g(x,y;z,z')=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty}\begin{pmatrix}\cos(n\alpha)\\ \sin(n\alpha)\end{pmatrix}\tilde{G}(k_\rho,z,z')\,e^{-jk_\rho\rho\cos(\alpha-\phi)}dk_\rho d\alpha$$

k_ρ, α complex plane

$$\boxed{g(x,y;z,z')=\frac{(1,-j)}{2\pi}\begin{pmatrix}\cos(n\phi)\\ \sin(n\phi)\end{pmatrix}\int_0^{\infty}\tilde{G}(k_\rho;z,z')\;J_n\left(k_\rho\rho\right)dk_\rho}$$

$\tilde{G}\left(k_\rho;z,z'\right)$ Spectral domain Green’s function directly from the z-transmission line

$J_n\left(k_\rho\rho\right)$ Bessel function of first species, n-order

SOLUTION TECHNIQUES FOR SOMMERFELD INTEGRALS

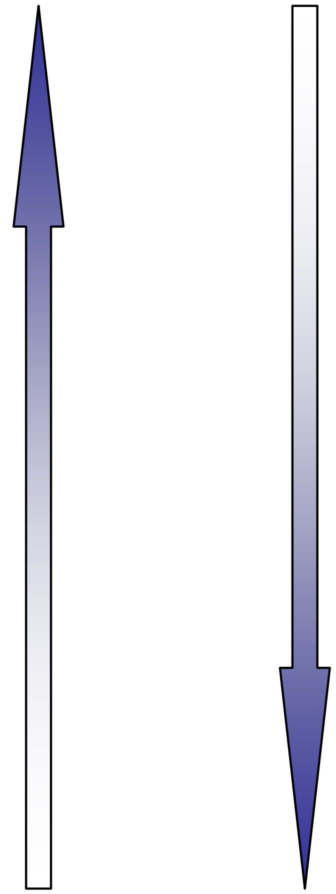
- Numerical integration on Sommerfeld contour
  - ☺ generality
  - ☹ computational cost

- Complex images
  - ☺ efficiency, distinguished physical contributions
  - ☹ lack of generality for increasing complexity

- Numerical integration on Sommerfeld contour
  - ☺ efficiency, distinguished physical contributions
  - ☹ topology of the integration plane

- **Asymptotic evaluation**
  - ☺ **extreme efficiency, physical insight**
  - ☹ **near zone deficiency, sensitive to the configuration**

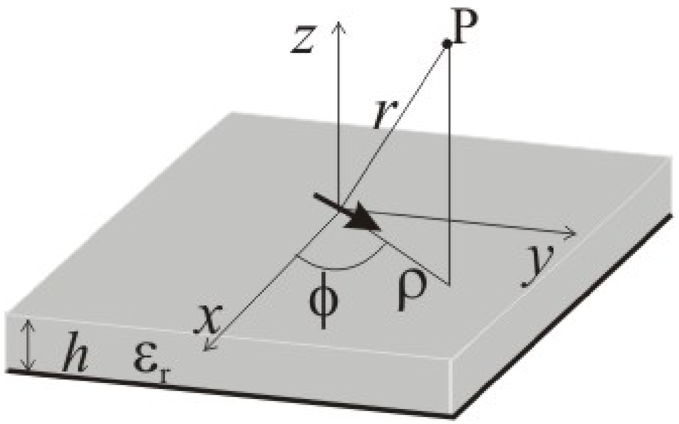
generality ***efficiency***

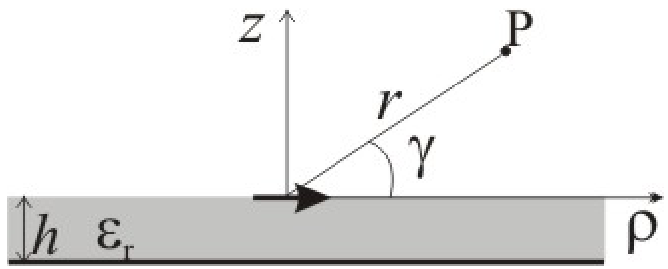


ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

Assumptions:

1. horizontal dipole at the air-dielectric interface
2. single layer (no-loss) dielectric slab
3. observation in free-space



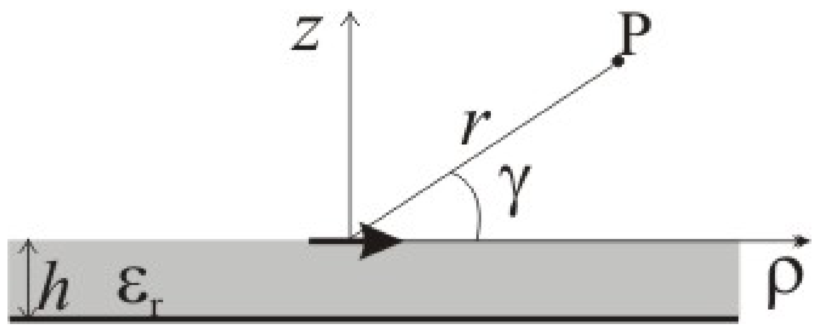


$$g(x,y,z)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\tilde{G}(k_\rho)\ H_0^2\left(k_\rho\rho\right)e^{-jk_z z}\,dk_\rho$$

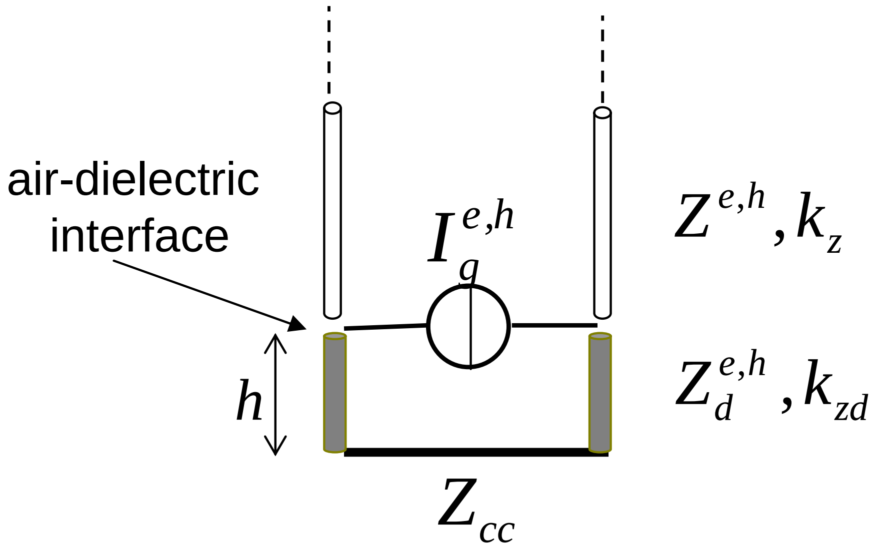
$H_0^2\left(k_\rho\rho\right)$ Hankel function of first species, second type

$\tilde{G}\left(k_\rho\right)=v^{e,h}\left(k_\rho\right),i^{e,h}\left(k_\rho\right)$ spectral voltage/current

ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS



Z-transmission line



$$v^e(k_\rho) = j\frac{\xi}{k}\frac{k_z k_{zd}\sin(k_{zd}h)}{D^e(k_\rho)}; \qquad v^h(k_\rho) = jk\xi\frac{\sin(k_{zd}h)}{D^h(k_\rho)}$$

$$i^e(k_\rho) = j\frac{k_{zd}\sin(k_{zd}h)}{D^e(k_\rho)}; \qquad i^h(k_\rho) = j\frac{k_z\sin(k_{zd}h)}{D^h(k_\rho)}$$

$$D^e(k_\rho) = \varepsilon_r k_z\cos(k_{zd}h) + jk_{zd}\sin(k_{zd}h); \qquad D^h(k_\rho) = k_{zd}\cos(k_{zd}h) + jk_z\sin(k_{zd}h)$$

$$k_{zd} = \sqrt{\varepsilon_r k^2 - k_\rho^2}$$

TOPOLOGY OF THE COMPLEX Kρ PLANE

$$g(x,y,z)=\frac{1}{4\pi}\int_C \tilde{G}(k_\rho)H_0^2\left(k_\rho \rho\right)e^{-jk_z z}dk_\rho$$

Branch cuts:

- sign of $k_z=\sqrt{k^2-k_\rho^2}$
- Hankel function

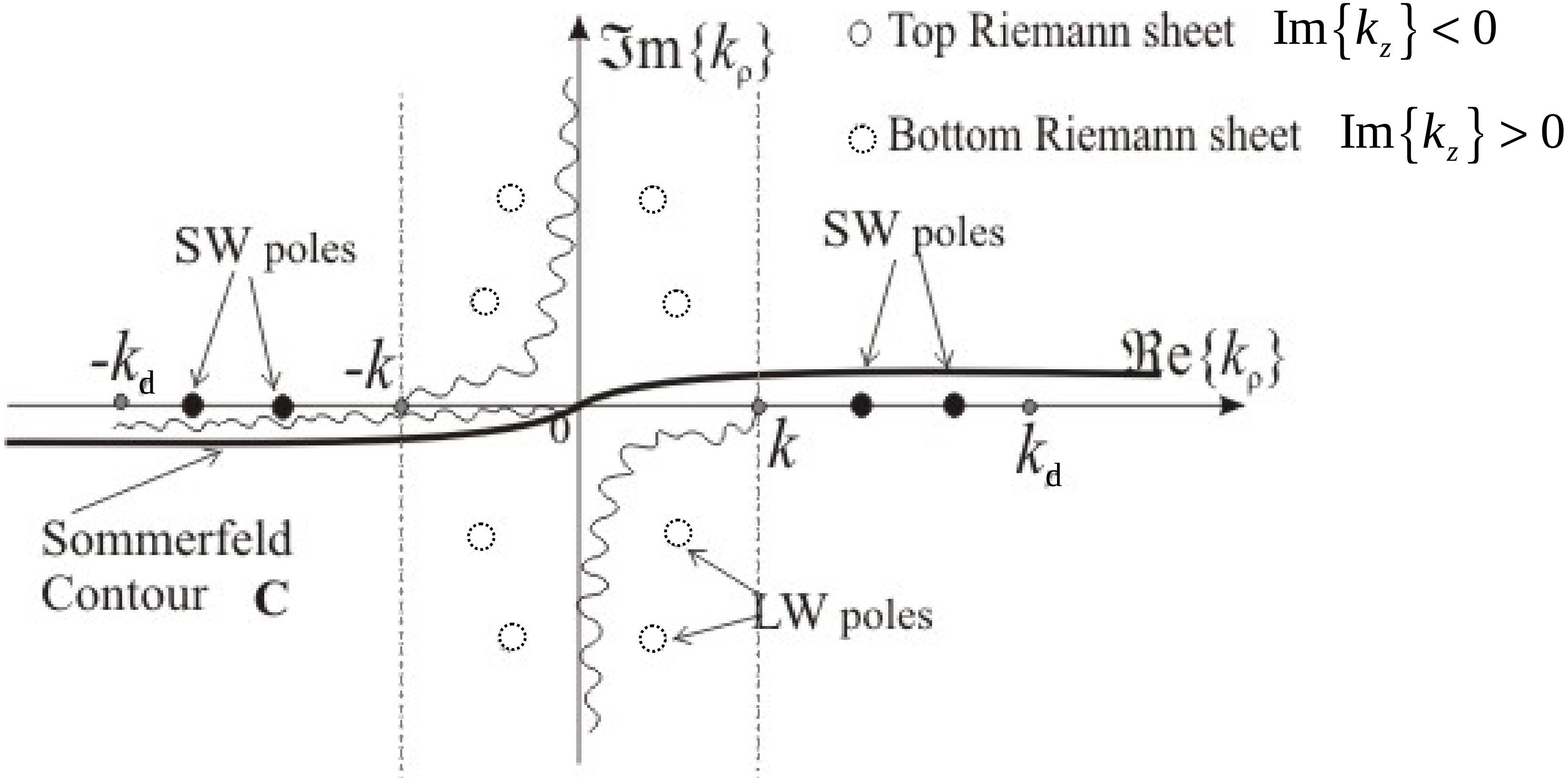
Poles (guided):

- SW poles k_ρ^{SW}
- LW poles k_ρ^{LW}

zeros of

$D^e\left(k_\rho\right)=\varepsilon_r k_z \cos\left(k_{zd}h\right)+jk_{zd}\sin\left(k_{zd}h\right);$

$D^h\left(k_\rho\right)=k_{zd}\cos\left(k_{zd}h\right)+jk_z\sin\left(k_{zd}h\right)$



ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

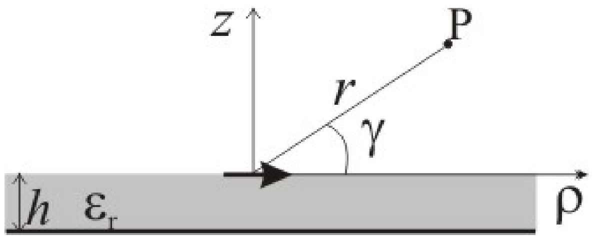
1. Large argument expression for Hankel function

$$g(x,y,z)=\frac{1}{4\pi}\int_C \tilde{G}(k_\rho)H_0^2\left(k_\rho\rho\right)e^{-jk_z z}dk_\rho=\frac{1}{4\pi}\sqrt{\frac{2j}{\pi}}\int_C \tilde{G}(k_\rho)\frac{e^{-jk_\rho\rho}}{\sqrt{k_\rho\rho}}e^{-jk_z z}dk_\rho$$

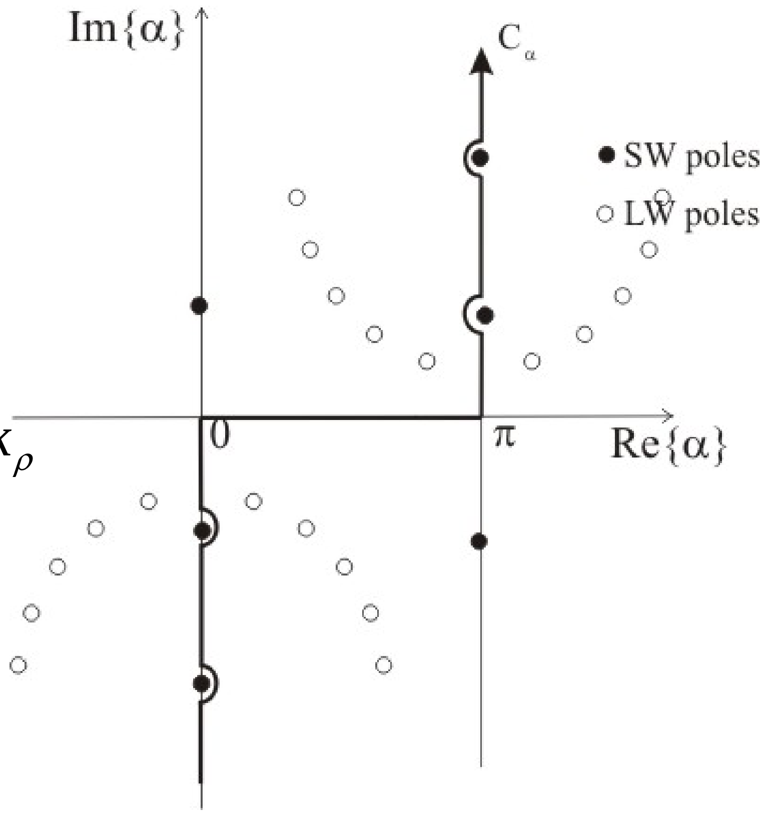
highly oscillatory in (-∞,∞)

2. Convenient change of variables in the complex angular plane

$$k_\rho=k\cos\alpha \qquad \rho=r\cos\gamma$$
$$k_z=k\sin\alpha \qquad z=r\sin\gamma$$



$$g(x,y,z)=\frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\int_{C_\alpha}\tilde{G}(\alpha)\frac{\sin\alpha}{\sqrt{\cos\alpha\cos\gamma}}\frac{e^{-jkr\cos(\alpha-\gamma)}}{\sqrt{kr}}dk_\rho$$



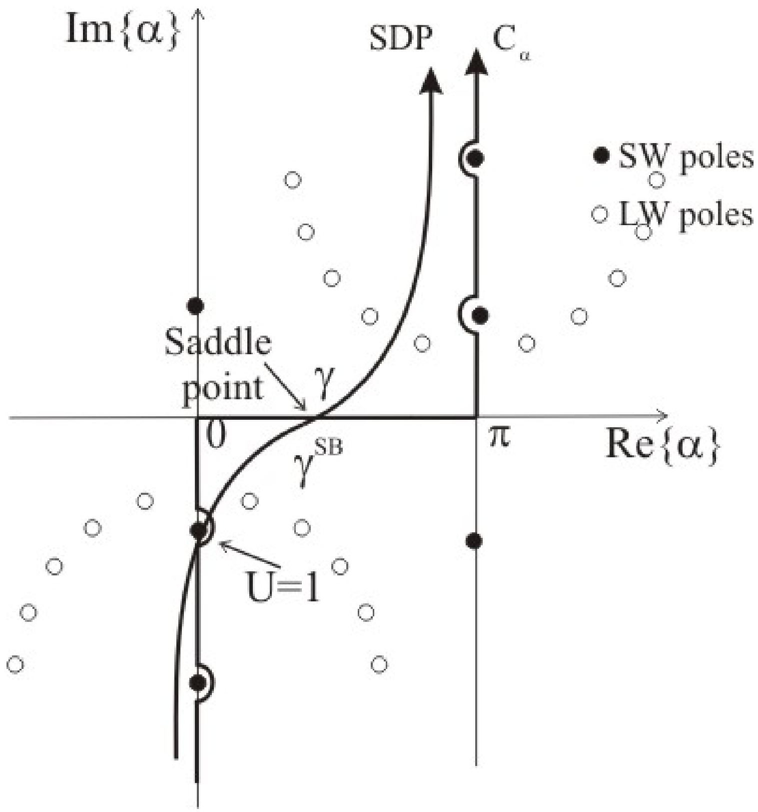
ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

3. Deformation onto the SDP

$$G^{SDP}(\alpha) = \tilde{G}(\alpha)\frac{\sin\alpha}{\sqrt{\cos\alpha\cos\gamma}}$$

$$g(x,y,z) = \frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\int_{SDP} G^{SDP}(\alpha)\frac{e^{-jkr\cos(\alpha-\gamma)}}{\sqrt{kr}}d\alpha +$$

$$\pm 2\pi j\frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\sum_{i=1}^{Np}\mathrm{Res}\left(\alpha_i^{S,LW}\right)U\left(\gamma_i^{SB}-\gamma\right)\frac{e^{-jkx\cos\left(\alpha_i^{S,LW}-\gamma\right)}}{\sqrt{kr}}$$



SDP: Steepest Descent Path

Res: residue of the integrand function in i-*th* S,LW pole

$$\mathrm{Res}\left(\alpha_i^{S,LW}\right) = \lim_{\alpha\to\alpha_i^{S,LW}}\left[\tilde{G}^{SDP}(\alpha)\right]$$

U: unit step function

γ_i^{SB}: Shadow Boundary (SB) angle

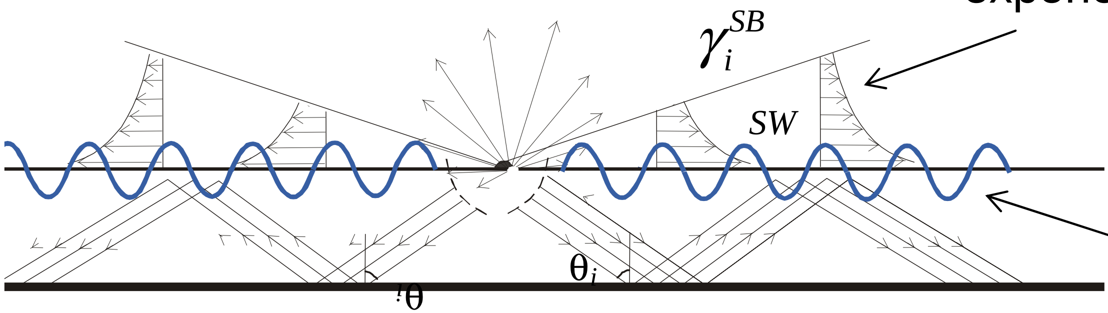
ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

Physical interpretation of residue contribution

$$g_i^{S,LW}(x,y,z) = \pm 2\pi j \frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\ \mathrm{Res}\left(\alpha_i^{S,LW}\right) U\left(\gamma_i^{SB} - \gamma\right)\frac{e^{-jkx\cos\left(\alpha_i^{S,LW}-\gamma\right)}}{\sqrt{kr}}$$

cylindrical wave

SW contribution



$\theta_i > \theta_{cr}$

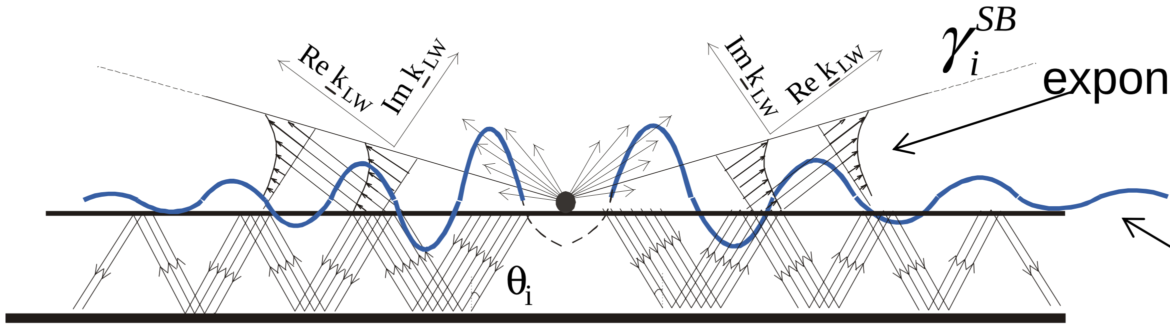
exponential decay along z

$k_\rho^{SW} > k$ slow wave

$$k_z = \sqrt{k^2 - k_\rho^{SW}} = -j\sqrt{k_\rho^{SW} - k^2}$$

no dumping along ρ

LW contribution



$\theta_i < \theta_{cr}$

exponential decay along $\mathrm{Im}\left\{\mathbf{k}^{LW}\right\}$

dumping along ρ

ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

4. Non uniform first order asymptotic evaluation of SDP integral

$$g^{SDP}(x,y,z)=\frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\frac{1}{\sqrt{kr}}\int_{SDP} G^{SDP}(\alpha)e^{-jkr\cos(\alpha-\phi)}d\alpha = A(k,r)\int_{SDP} f(z)e^{\Omega q(z)}dz$$

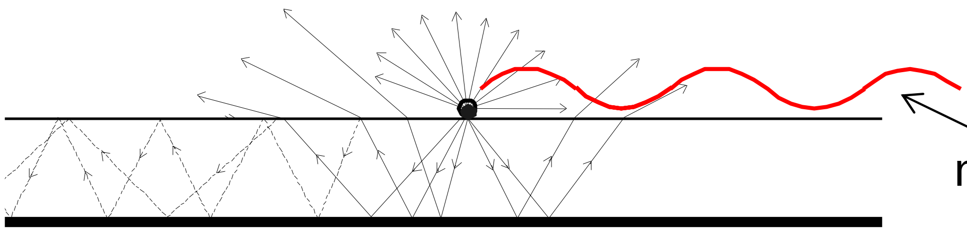
Asymptotic evaluation parameters

integrand	expansion argument	phase function	saddle point
$f(z)=G^{SDP}(\alpha);$	$\Omega = kr;$	$q(z)=-j\cos(\alpha-\phi)$	$q'(z)=j\sin(\alpha-\phi)=0;$ $z_s=\alpha_s=\gamma$

$$\boxed{g^{SDP}(x,y,z)\approx A(k,r)\sqrt{\frac{-2\pi}{\Omega q''(z_s)}}f(z_s)e^{\Omega q(z_s)} = \frac{kj}{2\pi}G^{SDP}(\gamma)\frac{e^{-jkr}}{kr}}$$

spherical wave

Space (SDP) contribution



no dumping along ρ

ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

Summary of the non-uniform first order asymptotic evaluation

$$g^{SDP}(x,y,z) \approx \frac{kj}{2\pi} G^{SDP}(\gamma) \frac{e^{-jkr}}{kr}$$

$$g_i^{S,LW}(x,y,z) = \pm \frac{kj}{2\pi} \sqrt{2\pi j}\ \mathrm{Res}\left(\alpha_i^{S,LW}\right) U\left(\gamma_i^{SB} - \gamma\right) \frac{e^{-jkx\cos\left(\alpha_i^{S,LW} - \gamma\right)}}{\sqrt{kr}}$$

$$\boxed{g(x,y,z) = g^{SDP}(x,y,z) + \sum_{i=1}^{Np} g_i^{S,LW}(x,y,z)}$$

- **discontinuous field**
- **affected by singularities**

ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

5. Uniform first order asymptotic evaluation of SDP integral

$$g^{SDP}(x,y,z) = \frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\frac{1}{\sqrt{kr}}\int_{SDP} G^{SDP}(\alpha)e^{-jkr\cos(\alpha-\gamma)}d\alpha$$

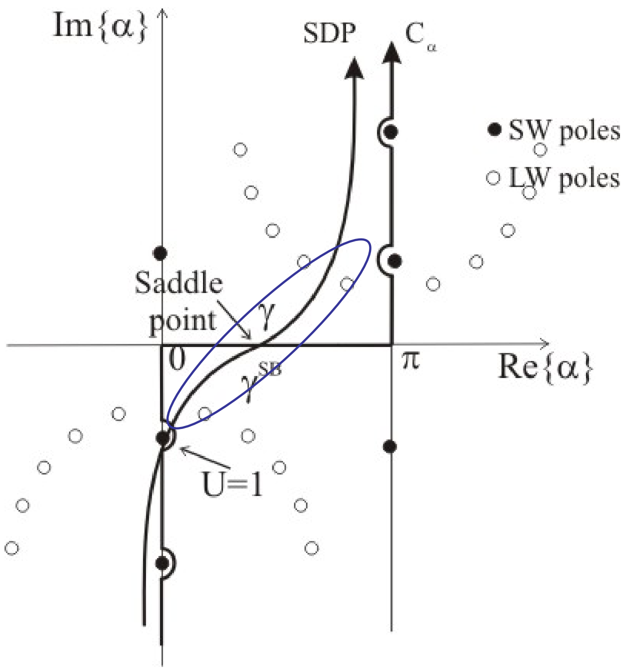
6. Wan der Waerden regularization technique

6a. $G^{SDP}(\alpha)$ has guided (SW,LW) poles

6b. Choose of $W(\alpha)$ with the same poles as $G^{SDP}(\alpha)$

$$W_i^{S,LW}(\alpha) = \frac{\mathrm{Res}\left(\alpha_i^{S,LW}\right)}{2\sin\left(\dfrac{\alpha-\alpha_i^{S,LW}}{2}\right)}$$

6c. Add and subtract the regularizing function



7. Regularization technique

$G^{reg}(\alpha)$

regular integrand → first order saddle point evaluation

$$g^{SDP}(x,y,z) = \frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\frac{1}{\sqrt{kr}}\int_{SDP}\left[G^{SDP}(\alpha) - \sum_i W_i^{S,LW}(\alpha)\right]e^{-jkr\cos(\alpha-\gamma)}d\alpha +$$

$$+\frac{k}{4\pi}\sqrt{\frac{2j}{\pi}}\frac{1}{\sqrt{kr}}\sum_i \int_{SDP} W_i^{S,LW}(\alpha)e^{-jkr\cos(\alpha-\gamma)}d\alpha$$

closed form

$$\sqrt{2\pi j}\,W_i^{S,LW}(\gamma)F(\mu_i^2)\frac{e^{-jkr}}{\sqrt{kr}}$$

$F(.)$: Fresnel transition function

$\mu_i^2 = 2kr\sin^2\left(\frac{\gamma-\alpha_i^{S,LW}}{2}\right)$: argument of the Fresnel transition function

ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

Summary of the uniform first order asymptotic evaluation

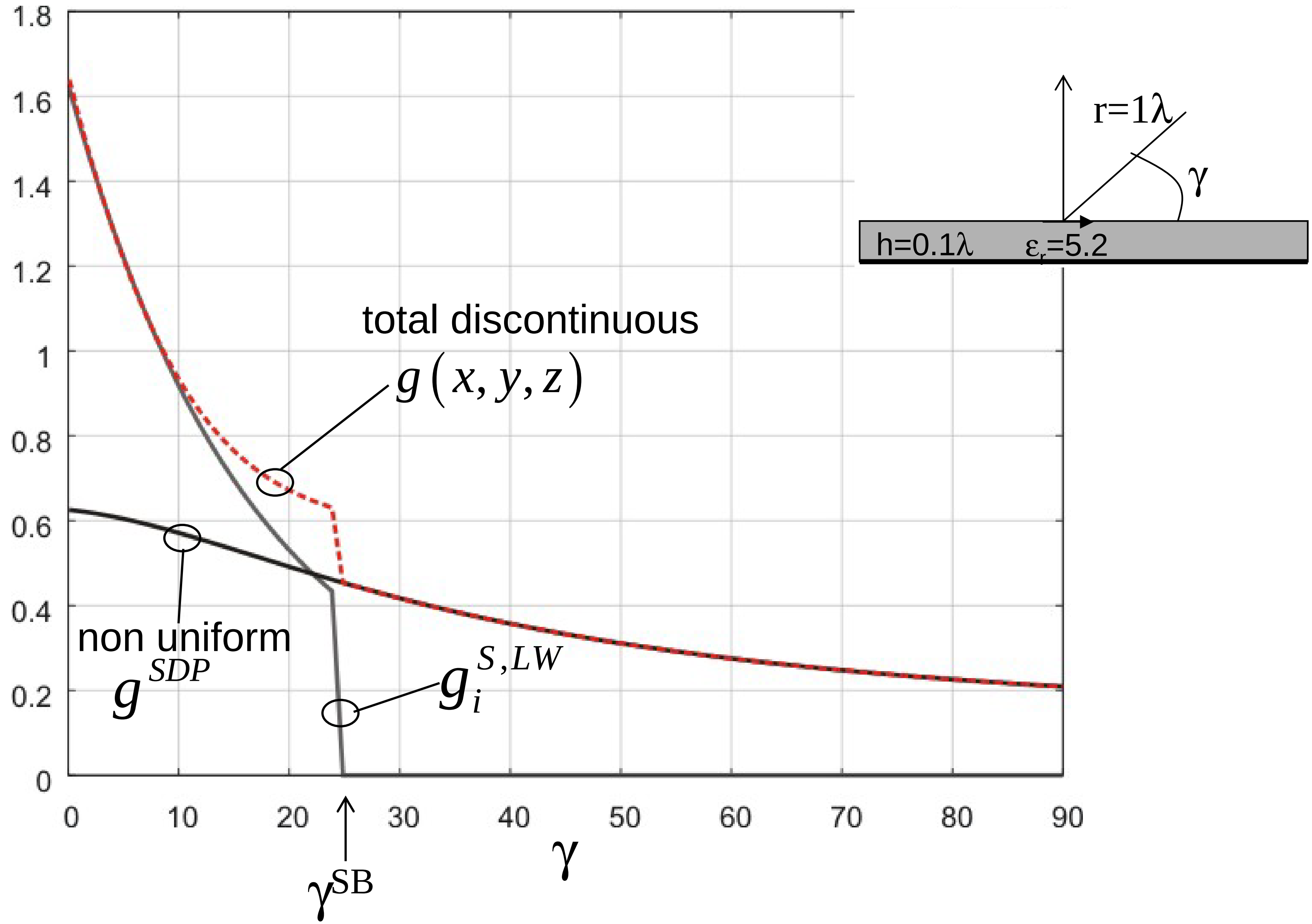
$$g^{SDP}(x,y,z) \approx \frac{kj}{2\pi} G^{reg}(\gamma) \frac{e^{-jkr}}{kr} + \frac{kj}{2\pi} \sum_{i=1}^{Np} W_i^{S,LW}(\gamma) F\left(\mu_i^2\right) \frac{e^{-jkr}}{kr}$$

$$g_i^{S,LW}(x,y,z) = \pm \frac{kj}{2\pi} \sqrt{2\pi j} \; \mathrm{Res}\left(\alpha_i^{S,LW}\right) U\left(\gamma_i^{SB} - \gamma\right) \frac{e^{-jkx\cos\left(\alpha_i^{S,LW} - \gamma\right)}}{\sqrt{kr}}$$

$$\boxed{g(x,y,z) = g^{SDP}(x,y,z) + \sum_{i=1}^{Np} g_i^{S,LW}(x,y,z)}$$

- **continuous field**
- **not affected by singularities**

An example of the compensation mechanism
Non uniform evaluation



ASYMPTOTIC EVALUATION OF SOMMERFELD INTEGRALS

An example of the compensation mechanism
Uniform evaluation