Multi-Resolution MoM for 3D structures

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Sparsify that!

Sparsification goal:
sparsify MoM matrix [Z] without altering accuracy of solution, $[Z]^{-1}$
Remember: sparsification=perturbation

$$\underbrace{Z}_{\text{actual}} = \underbrace{Z_{id}}_{\text{full}} + \underbrace{\delta Z}_{\text{Error, "perturbation"}}$$

Effect on solution:

$$\frac{\|\delta I\|}{\|I_{id}\|} \leq \text{const} \underbrace{K(Z)}_{\text{Condition number}} \frac{\|\delta Z\|}{\|Z_{id}\|}$$

Sparsify that!

Sparsification options:
1) Keep conditioning K constant, increase dynamics

$$\frac{\|\delta I\|}{\|I_{id}\|} \le \text{const}\; K(Z) \frac{\|\delta Z\|}{\|Z_{id}\|}$$

Lots of entries much smaller than diagonal: with a low clipping threshold achieve large sparsity, but with low relative error on [Z] mtx $\frac{\|\delta Z\|}{\|Z_{id}\|}$

⇨Use orthogonal transformations (basis change) (most wavelets and DWT do that)

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Sparsify that!

Sparsification options:
2) Improve conditioning K, increase dynamics if you can

$$\frac{\|\delta I\|}{\|I_{id}\|} \le \text{const}\; K(Z) \frac{\|\delta Z\|}{\|Z_{id}\|}$$

Good conditioning keep solution error small even with not-so low clipping threshold

⇨Do not use orthogonal transformation (basis change)

If you fix the conditioning problem, you also improve the convergence of iterative solvers
-Convergence speed related to eigenvalue spectrum
-For a given solution error you can use a higher residual error threshold

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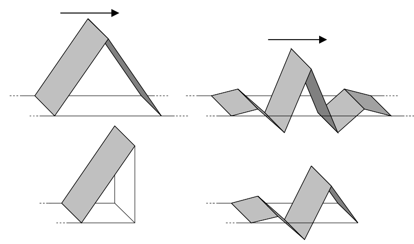
The moment-killing story

Rooftops | Wavelets



Non-zero average (0^{th} moment) | Zero average (0^{th} moment)

$$\underline{\mu}_m^0 = \int_{S_m} d\Sigma \, \underline{f}_m$$

Graphics: courtesy R. Loison

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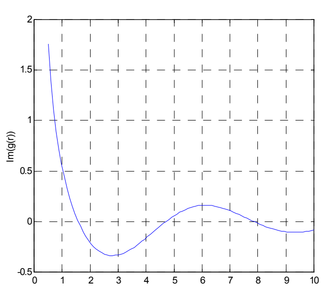
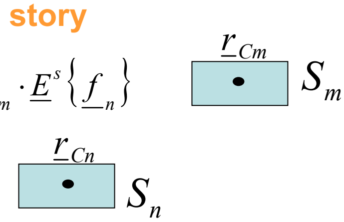
The moment-killing story



$$Z_{mn} = \int_{S_m} d\Sigma \, \underline{f}_m \cdot \underline{E}^s \left\{ \underline{f}_n \right\}$$



Far-field is smooth

$$\underline{E}_n^s(\underline{r}) \approx \underline{E}_n^s(\underline{r}_{Cm}) + \text{const}(\underline{r} - \underline{r}_{Cm}) + \ldots$$

$$Z_{mn} = \underbrace{\underline{E}_n^s(\underline{r}_{Cn}) \cdot \int_{S_m} d\Sigma \, \underline{f}_m}_{=0} + \text{const} \int_{S_m} d\Sigma \, \underline{f}_m \cdot (\underline{r} - \underline{r}_{Cn}) + \ldots$$

for wavelets

$$\Rightarrow \quad Z_{mn} \approx 0$$

The multi-scale perspective

Solutions for antenna-array problems have **multiple scales** of variation

large scales (= slow):
global features
(e.g. standing waves)

fine scales (= fast):
near source points,
edges, junctions, etc.

Standard basis functions (RWG etc.) do **NOT** possess **different** scales

Multi-Resolution (MR) basis[1]

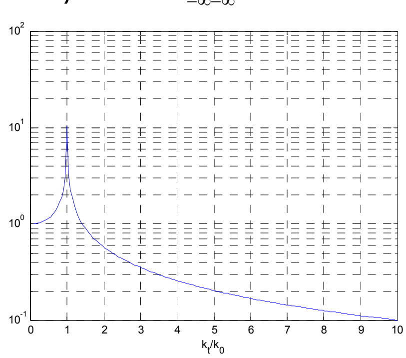
use representation of unknowns (currents) that keeps **different scales** on different subsets of basis functions

(1) F.Vipiana, P.Pirinoli, G.Vecchi, "A Multiresolution Method of Moments for Triangular Meshes", *IEEE Trans. Ant. Prop.*, Vol.53, No.7, July 2005.

The filtering properties of the Green's function

The Green's function is a (spatial) filter

$$\frac{\exp(-jk_0\rho)}{\rho} \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \, \exp(-j\underline{k}_t \cdot \underline{\rho}) \frac{1}{\sqrt{k_0^2 - k_t^2}}$$



Spectral representation of the Green's function

The filtering properties of the Green's function

The Green's function is a (spatial) filter

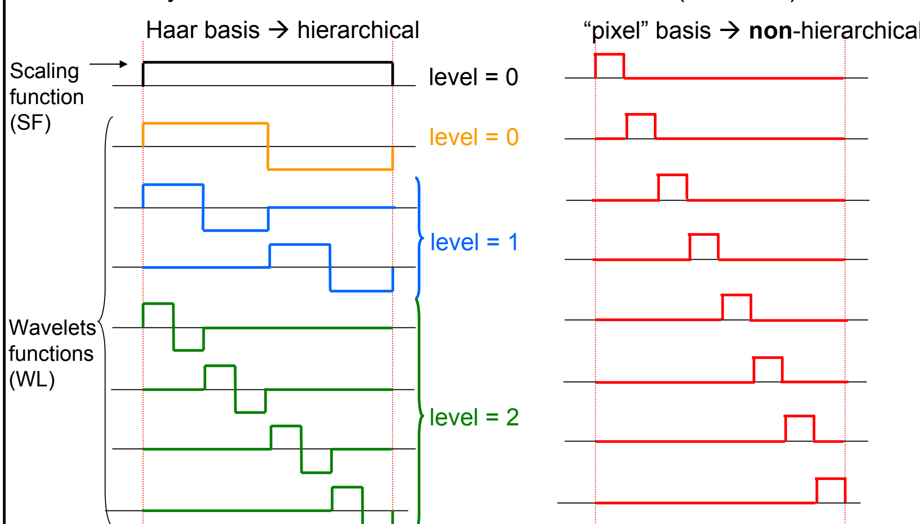
Basis functions with different spatial frequency content produce fields $\underline{E}^s\{\underline{f}_n\}$ with different strengths

Basis functions with different spatial frequency content produce diagonal entries of different magnitude

$$Z_{nn} = \int_{S_m} d\Sigma \underline{f}_n \cdot \underline{E}^s\{\underline{f}_n\}$$

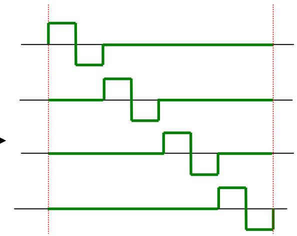
The dual-resolution of MR bases

- MR analysis uses a **hierarchical** basis function set (wavelets)

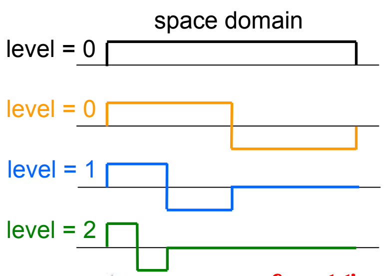


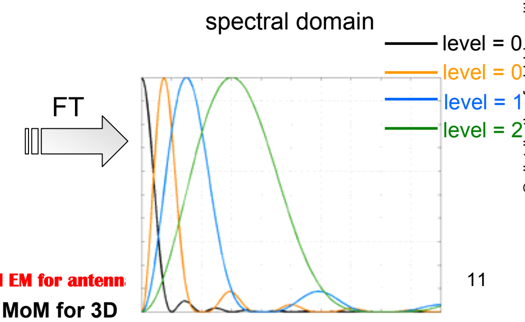
The dual-resolution of MR bases

- Wavelets have **resolution in space** (natural) domain:
  wavelets within a same level are shifted versions of the same function



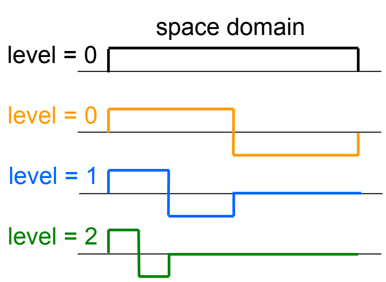
- Walets have **resolution in spectral** (conjugate) domain:
  wavelets of different levels have different occupation in the spectral domain

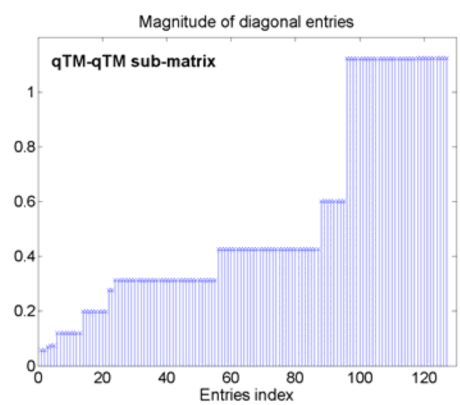




The filtering properties of the Green's function

MR basis:





MR functions of different levels have diagonal entries of different magnitude

Pixel basis functions have diagonal entries almost all equal

Diagonal preconditioning

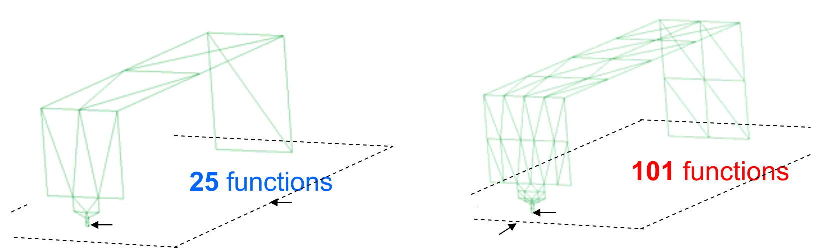
$$Z_{mn}^{PC} = \frac{Z_{mn}}{\sqrt{Z_{mm} Z_{nn}}}$$

Pixel basis (diagonal entries almost all equal): almost no effect

MR basis (diagonal entries have different magnitude)
Potential strong effect

Strategy: 1) adapt to meshed geometry

1) Wavelets of different levels have different level of detail: define them on cells of different size ⇨ construct a multi-scale system based on a family of different meshes of the same structure



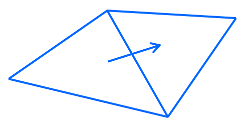
j=0 j=1

Strategy: 1) adapt to meshed geometry

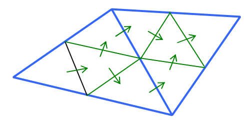
Each **j-level** MR function ⇒ linear combination of the **j-level** RWGs

Each **j-level** RWG ⇒ (exact reconstruction) linear combination of the **(j+1)-level** RWGs

j-level RWG



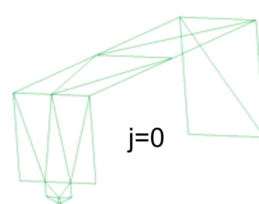
(j+1)-level RWGs



MR basis functions of **all** levels ⇒ linear combination of **RWG**s of the "**pixel** level" (last level)

Strategy: 1) adapt to meshed geometry

1) Handle coarsest mesh separately, use to define scaling functions (very different properties)



Resolution j | Coarse | Fine

$$V_j = V_0 \oplus \underbrace{W_0 \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_{j-1}}$$

Scaling functions *«Wavelets»*

- define a set of basis functions on coarse mesh
- non-unique choice
- Simplest: RWG of the mesh

Strategy: 2) handle scalar fields

On non-separable geometries, vector components are not separable (e.g. RWG)

Need a mapping from vector to scalar that "survives" on triangular grids…..

Then, define MR basis on scalar fields, and then come back to vector

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Mesh-independent scalar-vector mapping

Observe that

$$\nabla_s \cdot \underline{J}_{solenoidal}(\underline{r}) = 0 \quad \Longrightarrow \quad \underline{J}_{\text{solenoidal}}(\underline{r}) = \hat{n} \times \nabla_s S(\underline{r})$$

Scalar field

If we use the solenoidal extraction, the solenoidal part is mapped exactly onto a scalar field

But: we also need the non-solenoidal part (e.g. stars)…

$$\nabla_s \cdot \underline{J}_{\text{non-sol}}(\underline{r}) \neq 0$$

Can we map the non-solenoidal part to a scalar field?

The charge density! $\nabla_s \cdot \underline{J}_{\text{non-sol}}(\underline{r}) = -j\omega\sigma(\underline{r})$

(inversion charge → current is non-trivial; easier to understand on our wavelets: see later)

"Dirty" wavelets

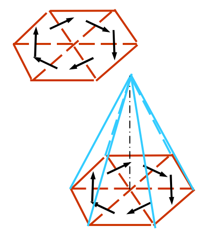
Haar wavelets are very simple and flexible, but can handle only piecewise-constant functions
Good for the charge, need extension to triangular mesh (easy)

Solenoidal part

$$\underline{J}_{\text{solenoidal}}(\underline{r}) = \hat{n} \times \nabla_s S(\underline{r})$$

RWG, rooftops

Piece-wise linear

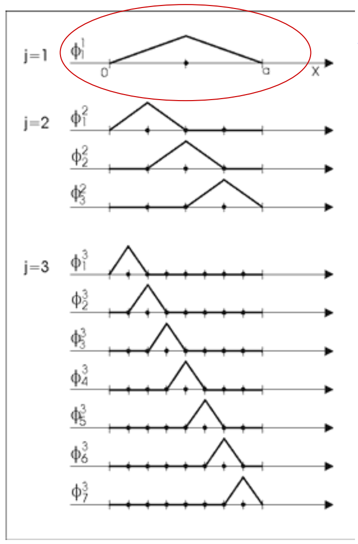


Need simple wavelets for this class

Lazy Wavelets (1)

- proposed by W. Sweldens in "The lifting scheme: A construction of second generation wavelets," *SIAM J. Mathematical Analysis*, vol. 29, no. 2, pp. 511–546, 1998.
- simply a subset of scaling functions
- improved by the "lifting scheme"
- adaptive to general shape domains (2D, non-separable)
- (very) low computational cost

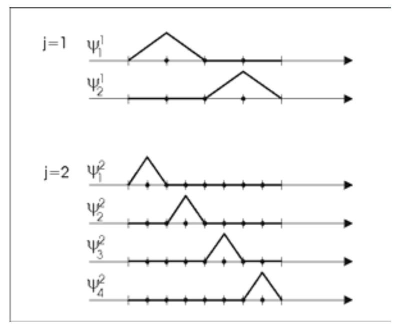
Lazy Wavelets (2)



Scaling functions

1D case

Wavelet functions

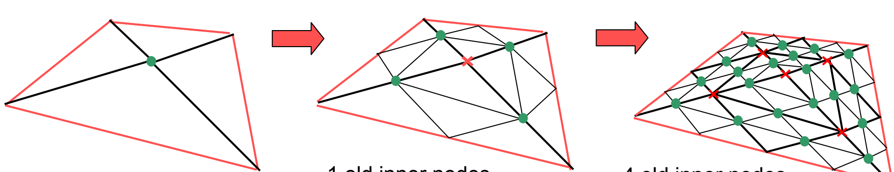


Generation of wavelets

solenoidal current, “TE” (loops) (node-based)

- generation of a sequence of multilevel **node** grids:
  - → each j-th level cell → divided in 4 (j+1)-th level cells
  - → ● = “**new**” inner nodes ✕ = “**old**” inner nodes

j = 1 level j = 2 level j = 3 level



1 new inner node

1 old inner nodes
4 new inner nodes

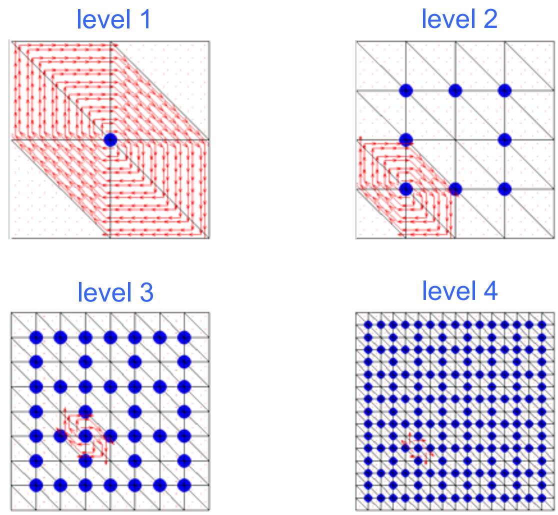
4 old inner nodes
20 new inner nodes

Lazy Wavelets: TE basis

● = selected inner nodes for each level

⟶ = TE vector function

2D case



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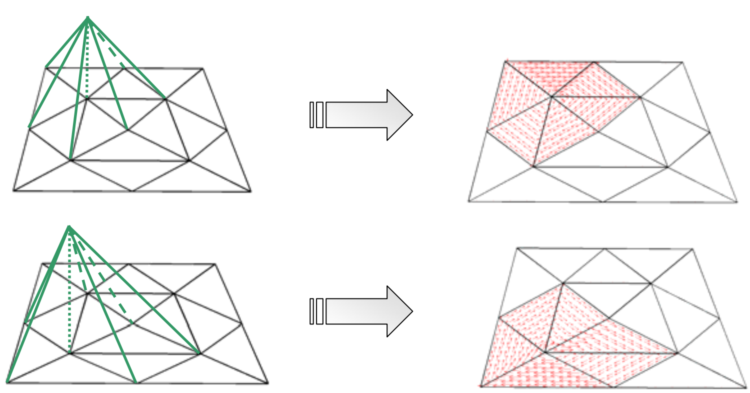
Lazy Wavelets: TE basis

- around each **new j-level** inner node generation of the TE functions:

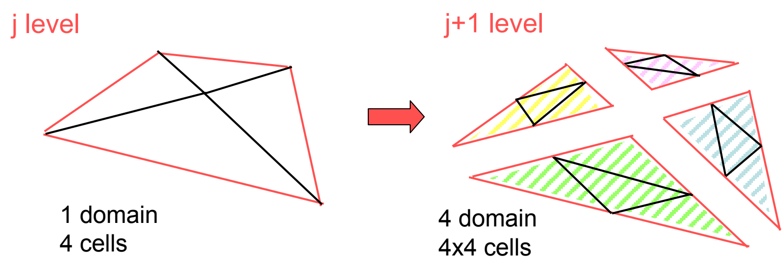
scalar solenoidal potential | **vector** current function



Generation of charge wavelets

Extension of Haar to triangular cells

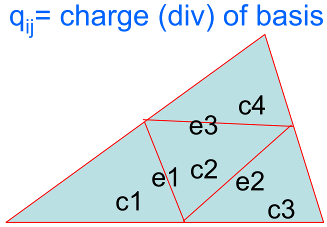
- generation of a sequence of multi-level **cell** grids:
  - → each j-th level cell → domain of the (j+1)-th level mesh
  - → on each (j+1)-th level domain → four-cells grid



Generation of charge wavelets

- on each **triangular** domain

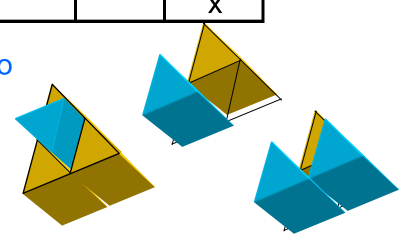
1. Construct "charge matrix"
q_{ij}= charge (div) of basis function #j (i.e. edge #j) into cell #i



	e1	e2	e3
c1	x		
c2	x	x	x
c3		x	
c4			x

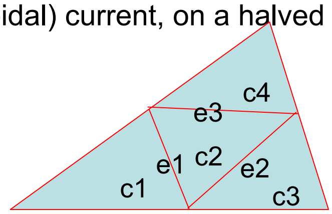
2. SVD([q]) defines the 3 non-zero independent "charge" states

$$[\psi_k], \quad k = 1,2,3$$



Mesh-independent scalar-vector mapping

From charge to (non-solenoidal) current, on a halved triangular cell



Recall

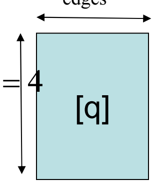
$$[q][I]=[c]$$

Current coefficients of J

Coefficients of div J, i.e. charge density distribution on mesh

$N_{\text{edges}}=3$

$N_{\text{cells}}=4$



$$[\Psi_k]=(\text{pinv}[q])[\psi_k]$$

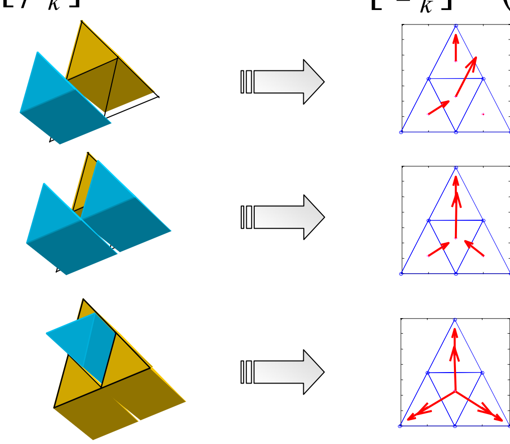
Mesh-independent scalar-vector mapping

scalar charge function

$$[\psi_k]$$

vector current function

$$[\Psi_k]=(\text{pinv}[q])[\psi_k]$$



"qTM" (quasi-irrotational) wavelets

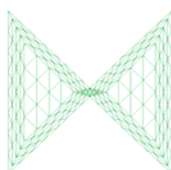
Generation of MR basis functions

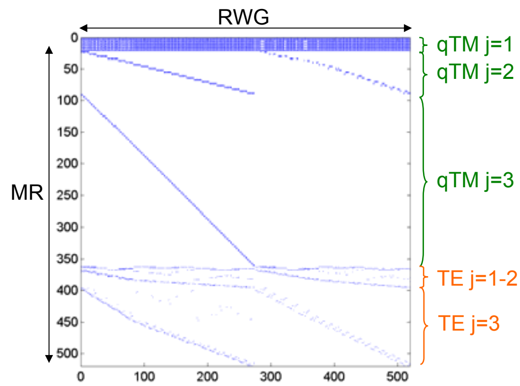
MR "basis change" matrix [f^{MR}] → efficient **preconditioner** of the RWG MoM matrix [Z]

↓

highly sparse matrix

No. of zero-elements = **94.5 %**





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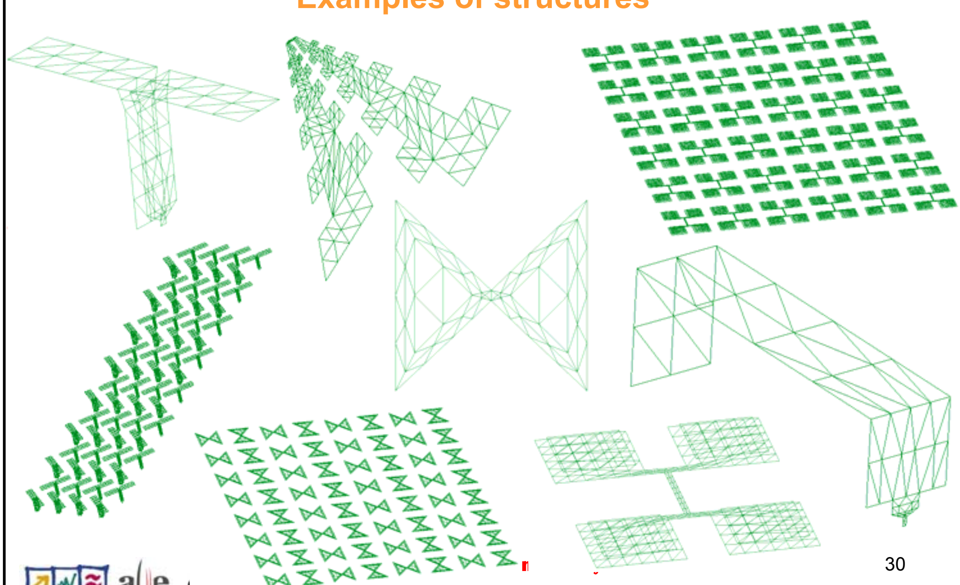
Examples of structures



Performance (1)

- **Condition number** of the [Z] matrix = $\frac{|\lambda_{\min}|}{|\lambda_{\max}|}$ λ_i = i-th eigvalue

- **Convergence**: number of iterations with the Conjugate Gradient[1] (CG) tolerance=10^{-4}

- % sparsification index: $\eta = \frac{\text{no. of matrix entries < threshold}}{\text{total no. of matrix entries}} \cdot 100$

- % **sparsification error** on the current $= \frac{\| [I] - [\tilde{I}] \|_2}{\| [I] \|_2} \cdot 100$

$[I]$ → solution with the **full** matrix $[\tilde{I}]$ → solution with **sparsified** matrix

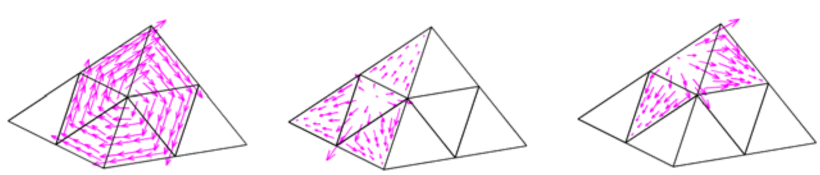
(1) *P. Sonneveld, "CGS: A fast Lanczos-type solver for nonsymmetric linear system", SIAM J. Sci. Stat. Comput.*, Vol. 10, No. 1, pp. 36-52, Jan. 1989.

Performance (2)

MR performace compared with:

- Rao-Wilton-Glisson (**RWG**) basis [1]
- Star-Loop (**SL**) basis [2][3]

after a **Diagonal Preconditioning** (DP) of the [Z] matrix



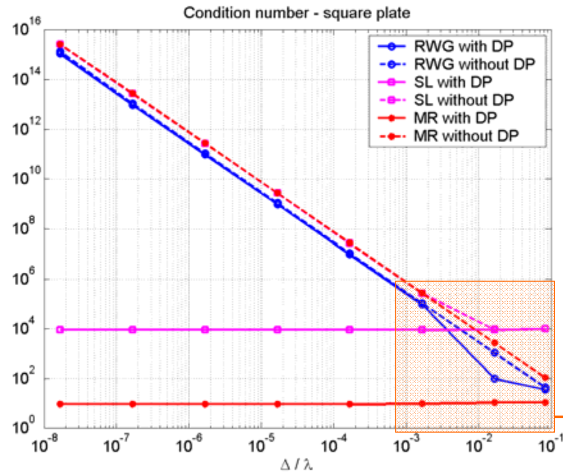
Loop function Star function Star function

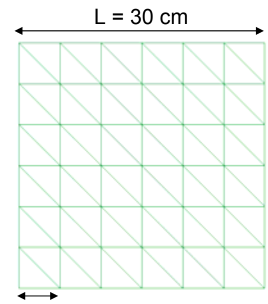
(1) S.M. Rao, D.R. Wilton, A.W. Glisson, "Electromagnetic Scattering by Surface of Arbitrary Shape", *IEEE Trans. on Antennas and Propagation*, Vol.AP-30, No.3, pp.409-418, May 1982.

(2) G. Vecchi, "Loop-Star Decomposition of Basis Functions in the Discretization of the EFIE", *IEEE Trans. on Antennas and Propagation,* Vol. AP-47, No. 2, pp. 339-346, Feb. 1999.

(3) D.R. Wilton, "Topological considerations in surface patch and volume cell modeling of electromagnetic scatters", *URSI Int. Symp. Electromagn. Theory*, Santiago de Compostela (Spain), Aug. 1983, pp. 65-68.

Condition number (1)





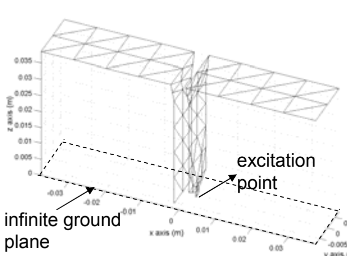
Δ = 5 cm

Square plate
96 functions

typical range for
antenna problems

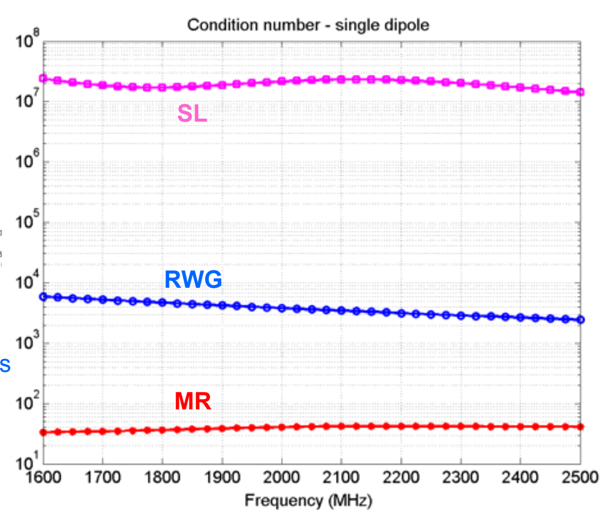
W.L. Wu, A.W. Glisson, D. Kajfez, "A study of two numerical solution procedures for the electric field integral equation at low frequency", ACES J., Nov. 1995, Vol. 10, No. 3.

Condition number (2)

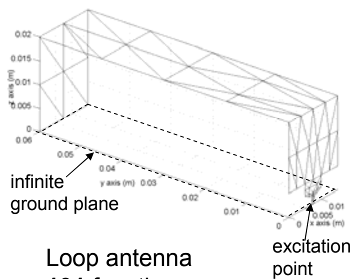


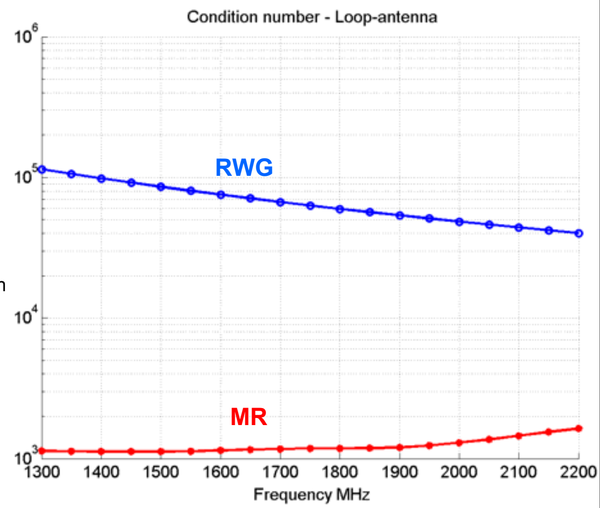
Dipole antenna
125 functions

RWG = Rao-Wilton-Glisson basis
MR = Multi-Resolution basis
SL = Star-Loop basis

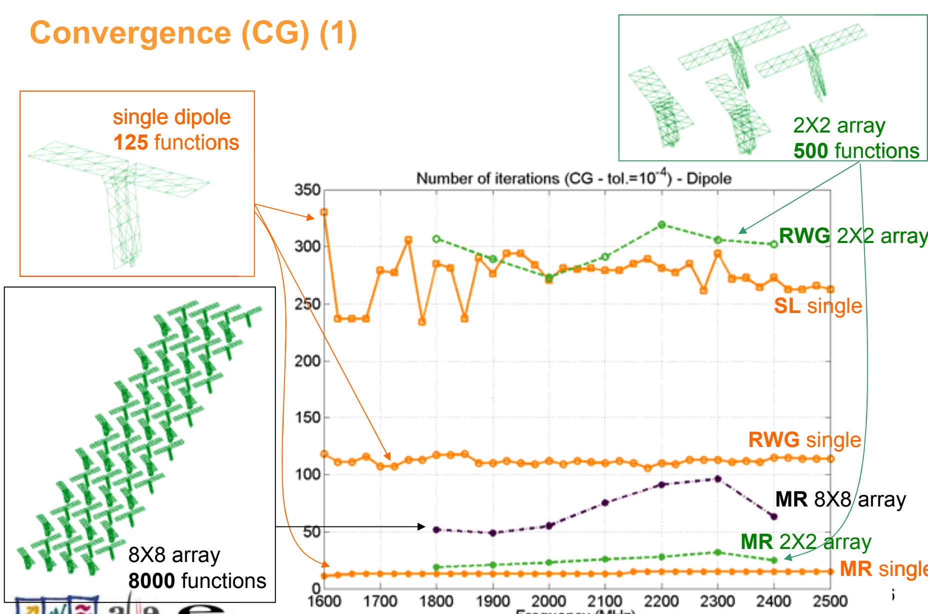


Condition number (3)

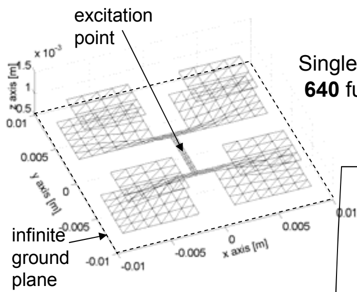




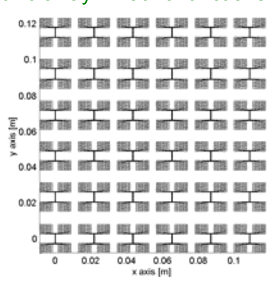
Convergence (CG) (1)

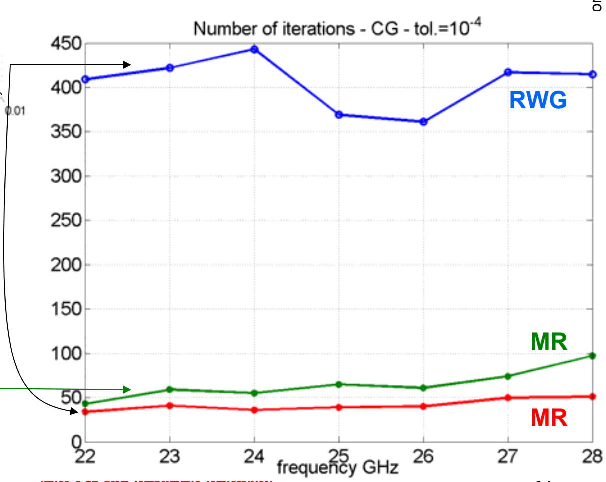


Convergence (CG) (2)



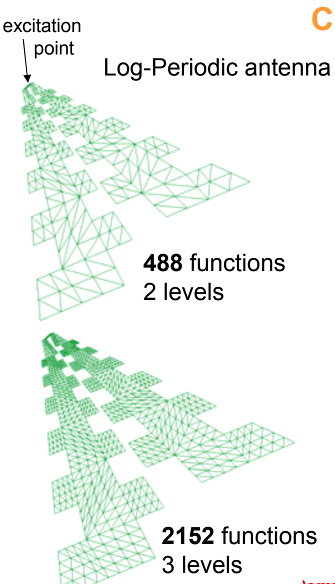
6X6 array – **23040** functions

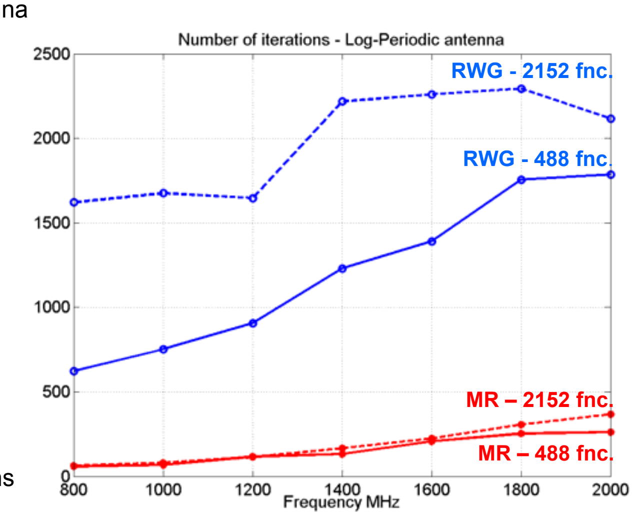




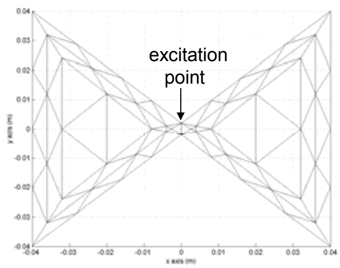
Convergence (CG) (3)

Log-Periodic antenna

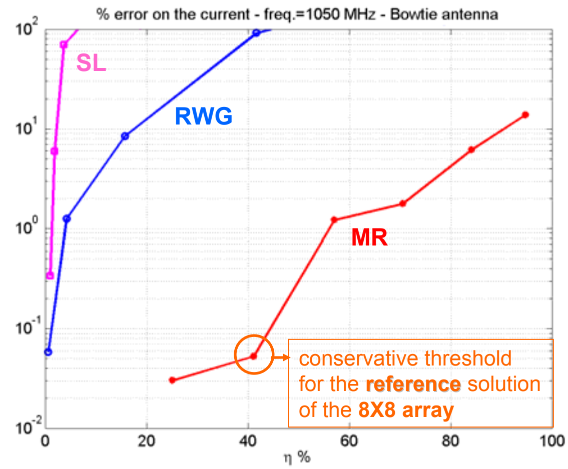




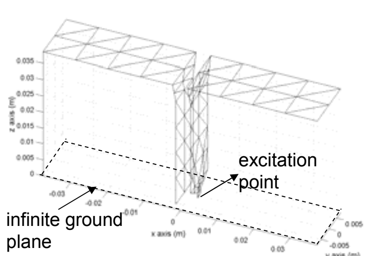
% Sparsification Error (1)



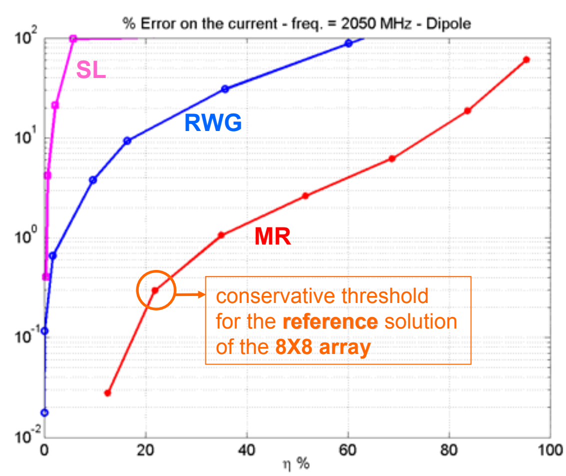
bowtie antenna
123 functions



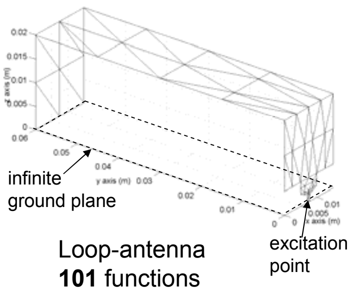
% Sparsification Error (2)



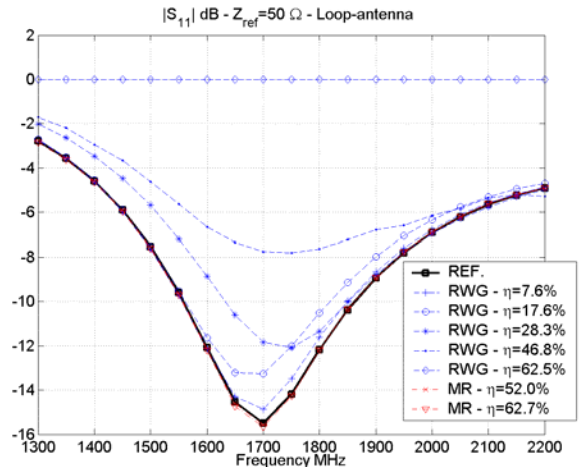
Single dipole
125 functions



S parameters (1)



Loop-antenna
101 functions



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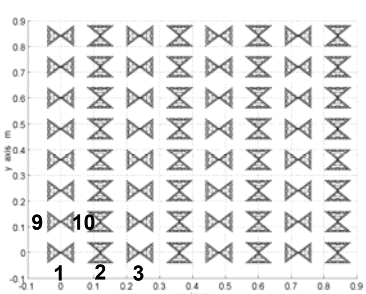
S parameters (2)

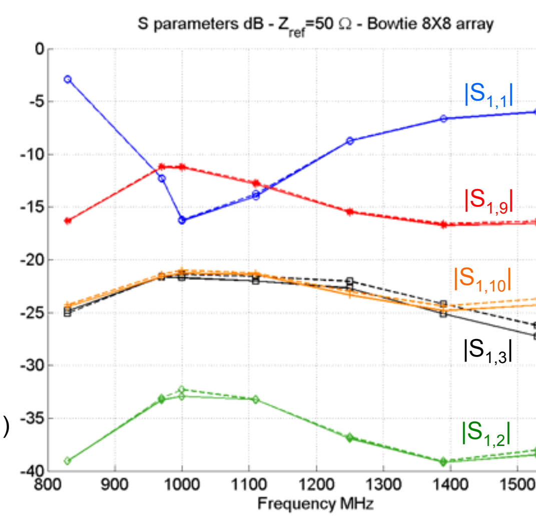


Bowtie 8X8 array
7872 functions

——— $\eta = 96.5\%$ (Reference)

- - - - $\eta = 99.0\%$

with **MR basis**



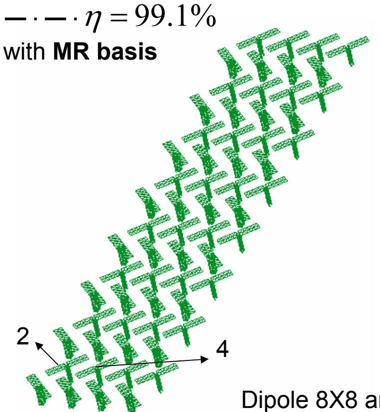
S parameters (3)

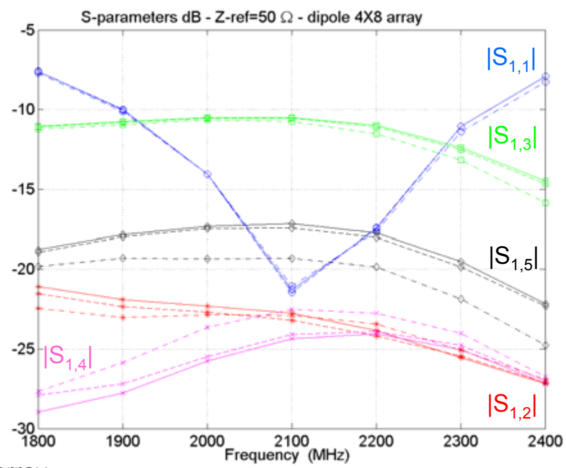
—— $\eta = 94.1\%$ (Reference)

- - - - $\eta = 97.6\%$

— · — · $\eta = 99.1\%$

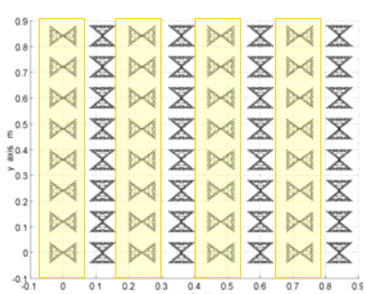
with **MR basis**

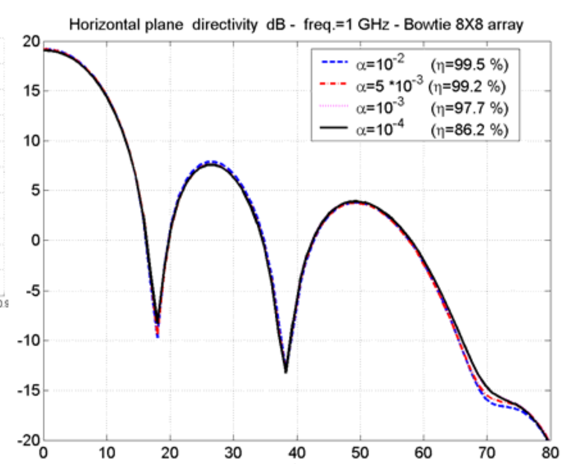




Dipole 8X8 array
8000 functions

Directvity





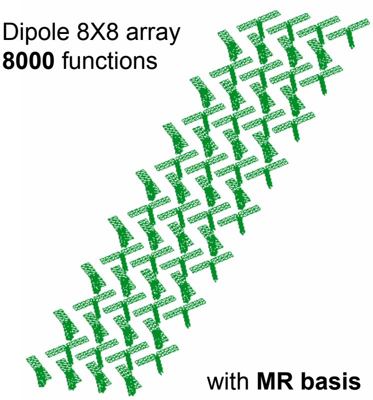
Bowtie 8X8 array
7872 functions

with **MR basis**

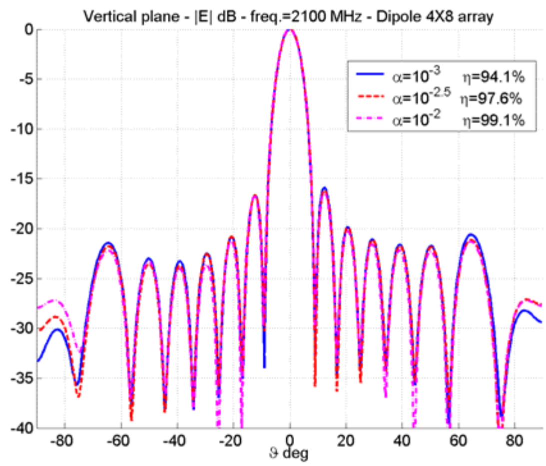
α = threshold

Radiation pattern

Dipole 8X8 array
8000 functions



with **MR basis**
$\alpha =$ threshold



Integration with other codes

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- Code example: the Banded Matrix Iterative Approach applied to a non-canonical grid by using the Adaptive Integral Method (**BMIA/AIM**) [1]
  (canonical grid = rectangular uniform grid)

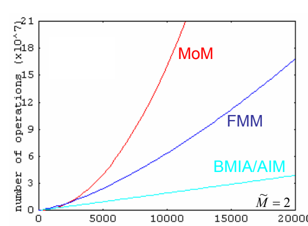
- Propeties of BMIA/AIM:

  BMIA/AIM $O\left(3(2\tilde{M}+1)N\log(2N)\right)$

  Standard MoM $O\left(N^2\right)$

  Single level FMM $O\left(15N^{3/2}\right)$

  N = number of RWG
  $\tilde{M}$ = Taylor Green function expansion order



- Basic limiting factor of BMIA/AIM: convergence of iterative solver

- Improvement: using the **MR** "basis change" matrix [f^{MR}] as an efficient **preconditioner** of BMIA/AIM.

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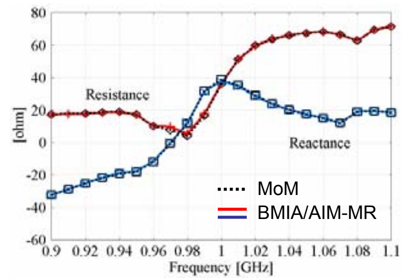
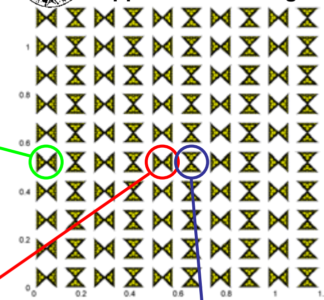
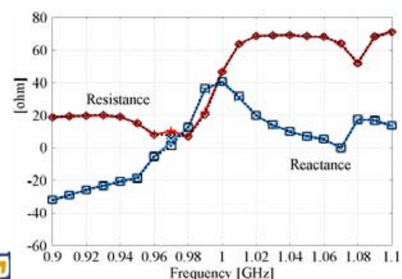
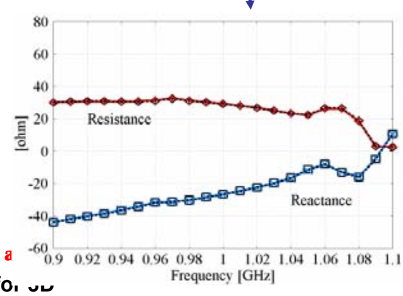
Integration BMIA/AIM – MR (1)









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Integration BMIA/AIM – MR (2)

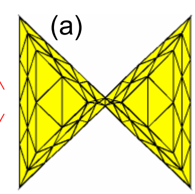
# of bow-ties pairs	antenna surface	Unk-nowns	$\mathbb{Z}^s$ non zero elements	FFT–2D points	$\mathbb{Z}^s$ filling time	MR basis - Prec MR CG iter. time	# iter.	Ram Mb
2×4	$2.16\,\lambda^2$	1,968	2,540,932	64×64	43″	0.55″	32	66
4×8	$9.61\,\lambda^2$	7,872	14,084,712	128×128	239″	2.82″	89	299
5×10	$15.34\,\lambda^2$	12,300	23,329,258	256×256	398″	5.59″	114	497
6×12	$22.09\,\lambda^2$	17,712	38,529,538	256×256	604″	7.69″	132	726
7×14	$29.88\,\lambda^2$	24,108	48,763,662	256×256	854″	10.20″	112	1002
8×16	$39.27\,\lambda^2$	31,488	64,953,520	256×256	1156″	13.46″	155	1479

# of bow-ties pairs	antenna surface	Unk-nowns	$\mathbb{Z}^s$ non zero elements	FFT–2D points	$\mathbb{Z}^s$ filling time	MR basis - Prec Diag CG iter. time	# iter.	Ram Mb
2×4	$2.16\,\lambda^2$	1,968	2,540,932	64×64	43″	0.42″	2925	62
4×8	$9.61\,\lambda^2$	7,872	14,084,712	128×128	239″	2.20″	3388	248
5×10	$15.34\,\lambda^2$	12,300	23,329,258	256×256	398″	4.60″	2157	409
6×12	$22.09\,\lambda^2$	17,712	38,529,538	256×256	604″	6.21″	3681	592
7×14	$29.88\,\lambda^2$	24,108	48,763,662	256×256	854″	8.20″	3354	809
8×16	$39.27\,\lambda^2$	31,488	64,953,520	256×256	1156″	10.34″	3190	1064

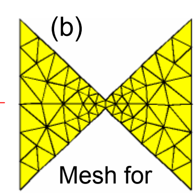
# of bow-ties pairs	antenna surface	Unk-nowns	$\mathbb{Z}^s$ non zero elements	FFT–2D points	$\mathbb{Z}^s$ filling time	Stand basis - Prec Diag CG iter. time	# iter.	Ram Mb
2×4	$2.16\,\lambda^2$	1,936	2,487,020	64×64	47″	0.42″	2251	61
4×8	$9.61\,\lambda^2$	7,744	13,806,956	128×128	263″	2.16″	2149	241
5×10	$15.34\,\lambda^2$	12,100	22,874,012	256×256	438″	4.55″	3358	397
6×12	$22.09\,\lambda^2$	17,424	34,212,460	256×256	663″	6.07″	3432	573
7×14	$29.88\,\lambda^2$	23,716	47,822,300	256×256	937″	8.08″	2372	786
8×16	$39.27\,\lambda^2$	30,976	63,703,532	256×256	1273″	11.00″	2742	1033

Mesh for BMIA/AIM-MR

(a)



(b)



Mesh for standard MoM

Taylor Green function expansion order = 3 (err. = 2%)
rd = 0.96 λ   Δ FFT = λ/20   frequency = 1 GHz

Integration BMIA/AIM – MR (3)

Summary of performance

Consider largest case, 8x16 pairs, 30,000 unknowns

BMIA with **standard RWG**

Total time[(1)]: 1273” (fill [Z]) +2742*11”=31,435”= **8h 44’**

BMIA with **MR**

Total time[(1)]: 1156” (fill [Z]) +155*14”=3,326”=**55’ 30’’**

(1) using a Pentium III – 700 MHz